On the renormalization of the sine–Gordon model

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Abstract

We analyse the renormalizability of the sine–Gordon model by the example of the two–point causal Green function up to second order in $\alpha_r(M^2)$, the dimensional coupling constant defined at the normalization scale M, and to all orders in β^2, the dimensionless coupling constant. We show that all divergences can be removed by the renormalization of the dimensional coupling constant using the renormalization constant Z_1, calculated in (J. Phys. A **36**, 7839 (2003)) within the path–integral approach. We show that after renormalization of the two–point Green function to first order in $\alpha_r(M^2)$ and to all orders in β^2 all higher order corrections in $\alpha_r(M^2)$ and arbitrary orders in β^2 can be expressed in terms of $\alpha_{\rm ph}$, the physical dimensional coupling constant independent on the normalization scale M. We calculate the Gell–Mann–Low function and analyse the dependence of the two–point Green function on $\alpha_{\rm ph}$ and the running coupling constant within the Callan–Symanzik equation. We analyse the renormalizability of Gaussian fluctuations around a soliton solution. We show that Gaussian fluctuations around a soliton solution are renormalized like quantum fluctuations around the trivial vacuum to first orders in $\alpha_r(M^2)$ and β^2 and do not introduce any singularity to the sine–Gordon model at $\beta^2 = 8\pi$. The finite correction to the soliton mass, coinciding with that calculated by Dashen *et al.* (Phys. Rev. D **10**, 4130 (1974)), appears in our approach to second order in $\alpha_{\rm ph}$ and to first order in β^2. This is a perturbative correction, which provides no singularity for the sine–Gordon model at $\beta^2 = 8\pi$. We calculate the correction to the soliton mass, caused by Gaussian fluctuations around a soliton, within the discretization procedure for various boundary conditions and find complete agreement with our result, obtained in continuous space–time.

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1 Introduction

The sine–Gordon model we describe by the Lagrangian [1, 2]

$$\mathcal{L}(x) = \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha_0(\Lambda^2)}{\beta^2}\,(\cos\beta\vartheta(x) - 1), \tag{1.1}$$

where the field $\vartheta(x)$ and the coupling constant β are unrenormalizable, $\alpha_0(\Lambda^2)$ is a dimensional *bare* coupling constant and Λ is an ultra–violet cut–off. As has been shown in [2] the coupling constant $\alpha_0(\Lambda^2)$ is multiplicatively renormalizable and the renormalized Lagrangian reads [2]

$$\begin{aligned}\mathcal{L}(x) &= \frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha_r(M^2)}{\beta^2}(\cos\beta\vartheta(x) - 1) + (Z_1 - 1)\frac{\alpha_r(M^2)}{\beta^2}(\cos\beta\vartheta(x) - 1) = \\ &= \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + Z_1\,\frac{\alpha_r(M^2)}{\beta^2}\,(\cos\beta\vartheta(x) - 1),\end{aligned} \tag{1.2}$$

where $Z_1 = Z_1(\alpha_r(M^2), \beta^2, M^2; \Lambda^2)$ is the renormalization constant [2]–[5] depending on the normalization scale M. The renormalization constant relates the renormalized coupling constant $\alpha_r(M^2)$, depending on the normalization scale M, to the *bare* coupling constant $\alpha_0(\Lambda^2)$ [2]–[5]:

$$\alpha_r(M^2) = Z_1^{-1}(\alpha_r(M^2), \beta^2, M^2; \Lambda^2)\,\alpha_0(\Lambda^2). \tag{1.3}$$

As has been found in [2] the renormalization constant $Z_1(\alpha_r(M^2), \beta^2, M^2; \Lambda)$ is equal to

$$Z_1(\alpha_r(M^2), \beta^2, M^2; \Lambda^2) = \Big(\frac{\Lambda^2}{M^2}\Big)^{\beta^2/8\pi}. \tag{1.4}$$

This result is valid to all orders of perturbation theory developed relative to the coupling constant β^2 and $\alpha_0(\Lambda^2)$ [2]. Since the normalization constant does not depend on $\alpha_r(M^2)$, we write below $Z_1 = Z_1(\beta^2, M^2; \Lambda^2)$.

For the analysis of the renormalizability of the sine–Gordon model with respect to quantum fluctuations around the trivial vacuum we expand the Lagrangian (1.2) in powers of $\vartheta(x)$. This gives

$$\mathcal{L}(x) = \frac{1}{2}\,[\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) - \alpha_r(M^2)\,\vartheta^2(x)] + \mathcal{L}_{\rm int}(x), \tag{1.5}$$

where $\mathcal{L}_{\rm int}(x)$ describes the self–interactions of the sine–Gordon field

$$\begin{aligned}\mathcal{L}_{\rm int}(x) &= \alpha_r(M^2)\sum^{\infty}_{n=2}\frac{(-1)^n}{(2n)!}\beta^{2(n-1)}\vartheta^{2n}(x) \\ &+ (Z_1 - 1)\,\alpha_r(M^2)\sum^{\infty}_{n=1}\frac{(-1)^n}{(2n)!}\beta^{2(n-1)}\vartheta^{2n}(x).\end{aligned} \tag{1.6}$$

It is seen that the coupling constant $\alpha_r(M^2)$ has the meaning of a squared mass of free quanta of the sine–Gordon field $\vartheta(x)$. The causal two–point Green function of free sine–Gordon quanta with mass $\alpha_r(M^2)$ is defined by

$$-\,i\,\Delta_F(x; \alpha_r(M^2)) = \langle 0|{\rm T}(\vartheta(x)\vartheta(0))|0\rangle = \int\frac{d^2k}{(2\pi)^2 i}\,\frac{e^{-i\,k\,\cdot\,x}}{\alpha_r(M^2) - k^2 - i\,0}. \tag{1.7}$$

At $x = 0$ the Green function $-i\,\Delta_F(0;\alpha_r(M^2))$ is equal to [2]

$$- i\,\Delta_F(0;\alpha_r(M^2)) = \frac{1}{4\pi}\,\ell n\Big[\frac{\Lambda^2}{\alpha_r(M^2)}\Big], \tag{1.8}$$

where Λ is a cut–off in Euclidean 2–dimensional momentum space [2].

The paper is organized as follows. In Section 2 we analyse the renormalizability of the sine–Gordon model by means of power counting. In Sections 3 we investigate the renormalizability of the two–point Green function of the sine–Gordon field. Quantum fluctuations are calculated relative to the trivial vacuum up to second order in $\alpha_r(M^2)$ and to all orders in β^2. We show that after renormalization of the two–point Green function to first order in $\alpha_r(M^2)$ and to all orders in β^2 all higher order corrections in $\alpha_r(M^2)$ and all orders in β^2 can be expressed in terms of $\alpha_{\rm ph}$, the physical dimensional coupling constant independent on the normalization scale M. We derive the effective Lagrangian of the sine–Gordon model, taking into account quantum fluctuations to second order in $\alpha_r(M^2)$ and to all orders in β^2. We show that the correction of second order in $\alpha_{\rm ph}$ and first order in β^2, i.e. $O(\alpha^2_{\rm ph}\beta^2)$, reproduces the finite contribution $-\sqrt{\alpha_{\rm ph}}/\pi$ to the soliton mass coinciding with that calculated by Dashen *et al.* [6, 7]. In our approach this is a perturbative correction, which does not lead to a singularity of the sine–Gordon model at $\beta^2 = 8\pi$. This confirms the absence of a singularity of the renormalized sine–Gordon model at $\beta^2 = 8\pi$ conjectured by Zamolodchikov and Zamolodchikov [8] and proved in [2]. In Section 4 we analyse the renormalizability of the sine–Gordon model within the Renormalization Group approach. We use the Callan–Symanzik equation for the derivation of the total two–point Green function of the sine–Gordon field in the momentum representation. We show that the two–point Green function depends on the running coupling constant $\alpha_r(p^2) = \alpha_{\rm ph}(p^2/\alpha_{\rm ph})^{\tilde{\beta}^2/8\pi}$, where $\tilde{\beta}^2 = \beta^2/(1 + \beta^2/8\pi) < 1$ for all β^2. In Section 5 we investigate the renormalizability of the sine–Gordon model with respect to Gaussian fluctuations around a soliton solution. We show that Gaussian fluctuations around a soliton solution lead to the same renormalized Lagrangian of the sine–Gordon model as quantum fluctuations around the trivial vacuum taken into account to first order in $\alpha_r(M^2)$ and β^2. In Section 6 we discuss the correction to the soliton mass induced by quantum fluctuations. We show that Gaussian fluctuations around a soliton solution reproduce the same correction as the quantum fluctuations around the trivial vacuum, calculated to first orders in $\alpha_r(M^2)$ and β^2. This correction does not contain the non–perturbative finite quantum correction obtained by Dashen *et al.* [6, 7]. In our analysis of the sine–Gordon model such a finite correction appears only to second order in $\alpha_{\rm ph}$ and to first order in β^2 (see Section 3). In Section 7 we discuss the calculation of the correction to the soliton mass ΔM_s, induced by Gaussian fluctuations, within a discretization procedure for various boundary conditions. We show that the result of the calculation of ΔM_s does not depend on the boundary conditions and agrees fully with that obtained in continuous space–time. In the Conclusion we summarize the obtained results and discuss them. In the Appendix we adduce the solutions of the differential equation related to the calculation of Gaussian fluctuations around a soliton.

2 Power counting and renormalization of the sine–Gordon model

As usual the general analysis of renormalizability of a quantum field theory is carried out in the form of *power counting*, the concept of the superficial degree of divergence of momentum integrals based on dimensional considerations [3]–[5].

The analysis of the convergence of a given Feynman diagram G within *power counting* is done by scaling all internal momenta with a common factor λ, $k_\ell \to \lambda k_\ell$, and looking at the behaviour $I_G \sim \lambda^{\omega(G)}$ at $\lambda \to \infty$. Since we deal with a quantum field theory of a (pseudo)scalar field, the propagator of such a field behaves as λ^{-2} at $\lambda \to \infty$.

Let a given Feynman diagram G contain L independent loops, I internal boson lines and V_{2n} vertices with $2n$ lines. Since we have no vertices with derivatives of the sine–Gordon field, the superficial degree of divergence $\omega(G)$ of a diagram G is [3]–[5]

$$\omega(G) = 2L - 2I. \tag{2.1}$$

The number of independent loops L is defined by [3]–[5]

$$L = I + 1 - \sum_{\{n\}} V_{2n}, \tag{2.2}$$

where the sum extends over all vertices defining the Feynman diagram G. Substituting (2.2) into (2.1) we can express the superficial degree of divergence $\omega(G)$ in terms of vertices only

$$\omega(G) = 2 - 2\sum_{\{n\}} V_{2n}. \tag{2.3}$$

This testifies the complete renormalizability of the sine–Gordon model.

Indeed, the vacuum energy density is quadratically divergent, since it corresponds to the “Feynman diagram without vertices”. Such a quadratic ultra–violet divergence of the vacuum energy density of the sine–Gordon model has been recently shown in [2]. Such a quadratic divergence of the vacuum energy density can be removed by normal–ordering the operator of the Hamilton density of the sine–Gordon model [2].

All Feynman diagrams with one vertex diverge logarithmically, and any Feynman diagram with more than one vertex converges. As has been shown in [2] by using the path–integral approach, all logarithmic divergences can be removed by the renormalization of the dimensional coupling constant $\alpha_0(\Lambda^2)$.

3 Renormalization of causal two–point Green function

The causal two–point Green function of the sine–Gordon field is defined by

$$-\,i\,\Delta(x) = \frac{1}{i}\frac{\delta}{\delta J(x)}\frac{1}{i}\frac{\delta}{\delta J(0)} Z[J]_{J=0}, \tag{3.1}$$

where $Z[J]$ is a generating functional of Green functions

$$Z[J] = \int \mathcal{D}\vartheta\, \exp\Big\{i\int d^2y\,[\mathcal{L}(y) + \vartheta(y)J(y)]\Big\} =$$
$$= \int \mathcal{D}\vartheta\, \exp\Big\{i\int d^2y\,\Big[\frac{1}{2}\Big(\partial_\mu\vartheta(y)\partial^\mu\vartheta(y) - \alpha_r(M^2)\,\vartheta^2(y)\Big) + \mathcal{L}_{\rm int}(y) + \vartheta(y)J(y)\Big]\Big\}, \quad (3.2)$$

normalized by $Z[0] = 1$, $J(x)$ is the external source of the sine–Gordon field $\vartheta(x)$.

Substituting (3.2) into (3.1) we get

$$\begin{aligned} -\,i\,\Delta(x) &= \int \mathcal{D}\vartheta\,\vartheta(x)\vartheta(0)\,\exp\Big\{i\int d^2y\,\mathcal{L}_{\rm int}(y)\Big\} \\ &\times\ \exp\Big\{\frac{i}{2}\int d^2y\,\Big[\partial_\mu\vartheta(y)\partial^\mu\vartheta(y) - \alpha_r(M^2)\,\vartheta^2(y)\Big]\Big\}, \end{aligned} \quad (3.3)$$

where $\mathcal{L}_{\rm int}(y)$ is given by (1.6). The r.h.s. of (3.3) can be rewritten in the form of a vacuum expectation value of a time–ordered product

$$-\,i\,\Delta(x) = \langle 0|{\rm T}\Big(\vartheta(x)\vartheta(0)\,\exp\Big\{i\int d^2y\ :\mathcal{L}_{\rm int}(y):\Big\}\Big)|0\rangle_c, \quad (3.4)$$

where the index c means the *connected* part, $:\ \ldots\ :$ denotes normal ordering, $\vartheta(x)$ is the free sine–Gordon field operator with mass $\alpha_r(M^2)$ and the causal two–point Green function $\Delta_F(x, \alpha_r(M^2))$ defined by (1.7).

In the momentum representation the two–point Green function (3.4) reads

$$\begin{aligned} -\,i\,\tilde{\Delta}(p) &= -i\int d^2x\, e^{+ip\,\cdot\,x}\Delta(x) = \\ &= \int d^2x\, e^{+ip\,\cdot\,x}\,\langle 0|{\rm T}\Big(\vartheta(x)\vartheta(0)\,\exp\Big\{i\int d^2y\ :\mathcal{L}_{\rm int}(y):\Big\}\Big)|0\rangle_c. \end{aligned} \quad (3.5)$$

For the analysis of the renormalizability of the sine–Gordon model we propose to calculate the corrections to the two–point Green function (3.4) (or to (3.5)), induced by quantum fluctuations around the trivial vacuum. Expanding the r.h.s. of Eq.(3.4) in powers of $\alpha_r(M^2)$ and β^2 we determine

$$-\,i\,\Delta(x) = \sum^{\infty}_{m=0}(-i)\,\Delta^{(m)}(x, \alpha_r(M^2)), \quad (3.6)$$

where $(-i)\,\Delta^{(m)}(x, \alpha_r(M^2))$ is defined by

$$-\,i\,\Delta^{(m)}(x, \alpha_r(M^2)) = \frac{i^m}{m!}\int\prod^m_{k=1} d^2y_k\langle 0|{\rm T}(\vartheta(x)\vartheta(0) : \mathcal{L}_{\rm int}(y_k)\,:)|0\rangle_c. \quad (3.7)$$

The Green function $(-i)\,\Delta^{(0)}(x, \alpha_r(M^2))$ coincides with the Green function (1.7) of the free sine–Gordon field.

In the momentum representation the correction to the two–point Green function $(-i)\,\Delta^{(m)}(x, \alpha_r(M^2))$ can be written as

$$\begin{aligned} -i\,\tilde{\Delta}^{(m)}(p, \alpha_r(M^2)) &= \int d^2x\, e^{+ip\,\cdot\,x}\,(-i)\,\Delta^{(m)}(x, \alpha_r(M^2)) = \\ = \frac{i^m}{m!}\int d^2x\, e^{+ip\,\cdot\,x} &\int\prod^m_{k=1} d^2y_k\langle 0|{\rm T}(\vartheta(x)\vartheta(0) : \mathcal{L}_{\rm int}(y_k)\,:)|0\rangle_c. \end{aligned} \quad (3.8)$$

The momentum representation is more convenient for the perturbative analysis of the renormalization of the two–point Green function of the sine–Gordon field.

3.1 Two–point Green function to first order in $\alpha_r(M^2)$ and to all orders in β^2

The correction to the two–point Green function to first order in $\alpha_r(M^2)$ and to all orders in β^2 is defined by

$$\begin{aligned}
&-i\,\Delta^{(1)}(x,\alpha_r(M^2)) = i\int d^2y_1\langle 0|\mathrm{T}(\vartheta(x)\vartheta(0) :\mathcal{L}_{\rm int}(y_1) :)|0\rangle_c = \\
&= i\,\alpha_r(M^2)\sum^{\infty}_{n=2}\frac{(-1)^n}{(2n)!}\beta^{2(n-1)}\int d^2y\,\langle 0|\mathrm{T}(\vartheta(x)\vartheta(0) : \vartheta^{2n}(y) :)|0\rangle_c \\
&+i\,\alpha_r(M^2)\,(Z_1-1)\sum^{\infty}_{n=1}\frac{(-1)^n}{(2n)!}\beta^{2(n-1)}\int d^2y\,\langle 0|\mathrm{T}(\vartheta(x)\vartheta(0) : \vartheta^{2n}(y) :)|0\rangle_c. \qquad (3.9)
\end{aligned}$$

Making all contractions we arrive at the expression

$$\begin{aligned}
&-i\,\Delta^{(1)}(x,\alpha_r(M^2)) = i\,\alpha_r(M^2)\sum^{\infty}_{n=2}\frac{(-1)^n}{(2n)!}\beta^{2(n-1)}[2n\,(2n-1)!!][-i\Delta_F(0,\alpha_r(M^2))]^{n-1} \\
&\times\int d^2y\,[-i\Delta_F(x-y,\alpha_r(M^2))][-i\Delta_F(-y,\alpha_r(M^2))] \\
&+i\,\alpha_r(M^2)\,(Z_1-1)\sum^{\infty}_{n=1}\frac{(-1)^n}{(2n)!}\beta^{2(n-1)}[2n\,(2n-1)!!][-i\Delta_F(0,\alpha_r(M^2))]^{n-1} \\
&\times\int d^2y\,[-i\Delta_F(x-y,\alpha_r(M^2))][-i\Delta_F(-y,\alpha_r(M^2))] \qquad (3.10)
\end{aligned}$$

The sums over n are equal to

$$\begin{aligned}
&\sum^{\infty}_{n=2}\frac{(-1)^n}{(2n)!}[2n\,(2n-1)!!][\beta^2(-i)\Delta_F(0,\alpha_r(M^2))]^{n-1} = 1-\exp\Big\{\frac{1}{2}\,\beta^2 i\Delta_F(0,\alpha_r(M^2))\Big\},\\
&\sum^{\infty}_{n=1}\frac{(-1)^n}{(2n)!}\beta^{2(n-1)}[2n\,(2n-1)!!][-i\Delta_F(0,\alpha_r(M^2))]^{n-1} = -\exp\Big\{\frac{1}{2}\,\beta^2 i\Delta_F(0,\alpha_r(M^2))\Big\}.
\end{aligned} \qquad (3.11)$$

Substituting (3.11) into (3.10) we get

$$\begin{aligned}
-\,i\,\Delta^{(1)}(x,\alpha_r(M^2)) &= i\,\alpha_r(M^2)\,\Big[1-\exp\Big\{\frac{1}{2}\,\beta^2 i\Delta_F(0,\alpha_r(M^2))\Big\}\Big]\\
&\times\int d^2y\,[-i\Delta_F(x-y,\alpha_r(M^2))][-i\Delta_F(-y,\alpha_r(M^2))]\\
&-\,i\,\alpha_r(M^2)\,(Z_1-1)\,\exp\Big\{\frac{1}{2}\,\beta^2 i\Delta_F(0,\alpha_r(M^2))\Big\}\\
&\times\int d^2y\,[-i\Delta_F(x-y,\alpha_r(M^2))][-i\Delta_F(-y,\alpha_r(M^2))]. \qquad (3.12)
\end{aligned}$$

The r.h.s. of (3.12) can be transcribed into the form

$$
\begin{aligned}
-\,i\,\Delta^{(1)}(x,\alpha_r(M^2)) &= i\,\alpha_r(M^2)\Big[1 - Z_1\,\exp\Big\{\frac{1}{2}\,\beta^2 i\Delta_F(0,\alpha_r(M^2))\Big\}\Big]\\
&\times \int d^2y\,[-i\Delta_F(x-y,\alpha_r(M^2))][-i\Delta_F(-y,\alpha_r(M^2))]. \quad (3.13)
\end{aligned}
$$

Using the normalization constant Z_1, given by (1.4), and the definition (1.8) of the two–point Green function we remove the cut–off Λ

$$
\begin{aligned}
-\,i\,\Delta^{(1)}(x,\alpha_r(M^2)) &= i\,\alpha_r(M^2)\Big[1 - \Big(\frac{\alpha_r(M^2)}{M^2}\Big)^{\beta^2/8\pi}\Big]\\
&\times \int d^2y\,[-i\Delta_F(x-y,\alpha_r(M^2))][-i\Delta_F(-y,\alpha_r(M^2))]. \quad (3.14)
\end{aligned}
$$

Thus, the renormalized causal two–point Green function of the sine–Gordon field, defined to first order in $\alpha_r(M^2)$ and to all orders in β^2 is given by

$$
\begin{aligned}
-\,i\,\Delta(x) &= -\,i\,\Delta_F(x,\alpha_r(M^2)) + i\,\alpha_r(M^2)\Big[1 - \Big(\frac{\alpha_r(M^2)}{M^2}\Big)^{\beta^2/8\pi}\Big]\\
&\times \int d^2y\,[-i\Delta_F(x-y,\alpha_r(M^2))][-i\Delta_F(-y,\alpha_r(M^2))]. \qquad (3.15)
\end{aligned}
$$

In the momentum representation the two–point Green function (3.15) reads

$$
-\,i\,\tilde{\Delta}(p) = \frac{(-i)}{\alpha_r(M^2)-p^2} + \frac{(-i)}{\alpha_r(M^2)-p^2}\,i\,\alpha_r(M^2)\Big[1 - \Big(\frac{\alpha_r(M^2)}{M^2}\Big)^{\beta^2/8\pi}\Big]\frac{(-i)}{\alpha_r(M^2)-p^2}. \tag{3.16}
$$

The second term defines the correction to the mass of the sine–Gordon field

$$
\delta\alpha_r(M^2) = -\,\alpha_r(M^2)\Big[1 - \Big(\frac{\alpha_r(M^2)}{M^2}\Big)^{\beta^2/8\pi}\Big]. \tag{3.17}
$$

Thus, the two–point Green function, calculated to first order in $\alpha_r(M^2)$ and to all orders in β^2 is equal to

$$
-\,i\,\tilde{\Delta}(p) = \frac{(-i)}{\alpha_r(M^2)+\delta\alpha_r(M^2)-p^2} = \frac{(-i)}{\alpha_{\rm ph}-p^2}, \tag{3.18}
$$

where $\alpha_{\rm ph}$ is determined by

$$
\alpha_{\rm ph} = \alpha_r(M^2) + \delta\alpha_r(M^2) = \alpha_r(M^2)\Big(\frac{\alpha_r(M^2)}{M^2}\Big)^{\beta^2/8\pi}. \tag{3.19}
$$

This gives also $\alpha_r(M^2)$ in term of M and $\alpha_{\rm ph}$:

$$
\alpha_r(M^2) = \alpha_{\rm ph}\Big(\frac{M^2}{\alpha_{\rm ph}}\Big)^{\tilde{\beta}^2/8\pi}\quad,\quad \tilde{\beta}^2 = \frac{\beta^2}{1+\dfrac{\beta^2}{8\pi}}. \tag{3.20}
$$

The Green function (3.18) can be obtained to leading order in β^2 from the Lagrangian

$$\begin{aligned}
\mathcal{L}_{\rm eff}(x) &= \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha_r(M^2)}{\beta^2}\Big(\frac{\alpha_r(M^2)}{M^2}\Big)^{\beta^2/8\pi}(\cos\beta\vartheta(x) - 1) = \\
&= \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha_{\rm ph}}{\beta^2}\,(\cos\beta\vartheta(x) - 1). \qquad (3.21)
\end{aligned}$$

We argue that higher order corrections to the two–point Green function in $\alpha_r(M^2)$ and to all orders in β^2 should depend on the physical coupling constant $\alpha_{\rm ph}$ only

$$\begin{aligned}
-\,i\,\Delta^{(m)}(x,\alpha_r(M^2)) &= \frac{i^m}{m!}\int\prod^m_{k=1}d^2y_k\langle 0|{\rm T}(\vartheta(x)\vartheta(0) : \mathcal{L}_{\rm int}(y_k) :)|0\rangle_c = \\
&= -i\,\Delta^{(m)}(x,\alpha_{\rm ph})\quad ({\rm for}\quad m\ge 2). \qquad (3.22)
\end{aligned}$$

In order to prove this assertion it is sufficient to analyse the renormalization of the causal two–point Green function to second order in $\alpha_r(M^2)$ and to all orders in β^2.

3.2 Two–point Green function to second order in $\alpha_r(M^2)$ and to all orders in β^2

The correction to the two–point Green function to second order in $\alpha_r(M^2)$ and to all orders in β^2 is defined by

$$\begin{aligned}
-\,i\,\Delta^{(2)}(x,\alpha_r(M^2)) &= -\,\frac{1}{2}\iint d^2y_1d^2y_2\,\langle 0|{\rm T}(\vartheta(x)\vartheta(0) : \mathcal{L}_{\rm int}(y_1) :: \mathcal{L}_{\rm int}(y_2) :)|0\rangle_c = \\
&= -\,\frac{1}{2}\,\alpha^2_r(M^2)\sum^\infty_{n_1=2}\frac{(-1)^{n_1}}{(2n_1)!}\beta^{2(n_1-1)}\sum^\infty_{n_2=2}\frac{(-1)^{n_2}}{(2n_2)!}\beta^{2(n_2-1)} \\
&\times \iint d^2y_1d^2y_2\,\langle 0|{\rm T}(\vartheta(x)\vartheta(0) : \vartheta^{2n_1}(y_1) :: \vartheta^{2n_2}(y_2) :)|0\rangle_c \\
&- \alpha^2_r(M^2)(Z_1 - 1)\sum^\infty_{n_1=2}\frac{(-1)^{n_1}}{(2n_1)!}\beta^{2(n_1-1)}\sum^\infty_{n_2=1}\frac{(-1)^{n_2}}{(2n_2)!}\beta^{2(n_2-1)} \\
&\times \iint d^2y_1d^2y_2\,\langle 0|{\rm T}(\vartheta(x)\vartheta(0) : \vartheta^{2n_1}(y_1) :: \vartheta^{2n_2}(y_2) :)|0\rangle_c \\
&- \frac{1}{2}\,\alpha^2_r(M^2)(Z_1 - 1)^2\sum^\infty_{n_1=1}\frac{(-1)^{n_1}}{(2n_1)!}\beta^{2(n_1-1)}\sum^\infty_{n_2=1}\frac{(-1)^{n_2}}{(2n_2)!}\beta^{2(n_2-1)} \\
&\times \iint d^2y_1d^2y_2\,\langle 0|{\rm T}(\vartheta(x)\vartheta(0) : \vartheta^{2n_1}(y_1) :: \vartheta^{2n_2}(y_2) :)|0\rangle_c. \qquad (3.23)
\end{aligned}$$

This expression can be described in terms of two classes of topologically different Feynman diagrams depicted in Fig.1. In the momentum representation this correction is equal to

$$-i\,\tilde{\Delta}^{(2)}(p,\alpha_r(M^2)) = i\,\Big[\alpha_r(M^2)Z_1\exp\Big\{\frac{1}{2}\,\beta^2 i\Delta_F(0,\alpha_r(M^2))\Big\}\Big]^2\,\Big[\frac{(-i)}{\alpha_r(M^2) - p^2}\Big]^2$$

$$\times\Big\{\sum^\infty_{n=1}\beta^{4n}\int\frac{d^2q_1}{(2\pi)^2 i}\,\frac{1}{\alpha_r(M^2) - q^2_1}\int\frac{d^2q_2}{(2\pi)^2 i}\,\frac{1}{\alpha_r(M^2) - q^2_2}\ldots$$

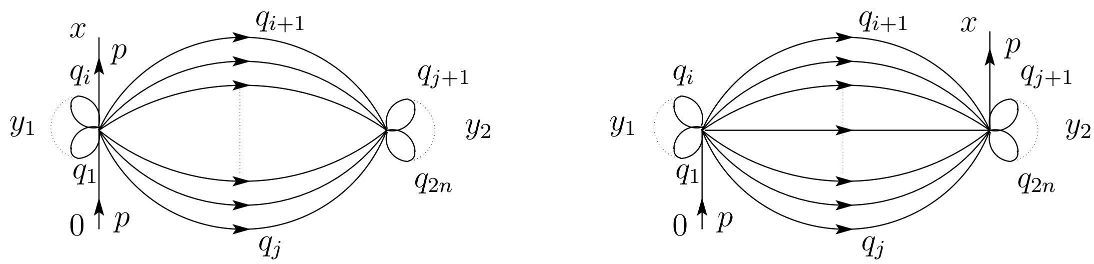

Figure 1: Feynman diagrams for corrections to the two–point Green function to second order in α and to arbitrary order in β^2. The left diagrams correspond to a non–vanishing expectation value for an even number of internal lines between the two vertices only, while the right diagrams describe a non–vanishing contribution for an odd number of internal lines between two vertices.

$$\times\int\frac{d^2q_{2n}}{(2\pi)^2i}\,\frac{1}{\alpha_r(M^2)-q^2_{2n}}\,\frac{1}{\alpha_r(M^2)-(p-q_1-q_2-\ldots-q_{2n})^2}$$
$$+\sum^{\infty}_{n=0}\beta^{4n+2}\int\frac{d^2q_1}{(2\pi)^2i}\,\frac{1}{\alpha_r(M^2)-q^2_1}\int\frac{d^2q_2}{(2\pi)^2i}\,\frac{1}{\alpha_r(M^2)-q^2_2}\ldots$$
$$\times\int\frac{d^2q_{2n+1}}{(2\pi)^2i}\,\frac{1}{\alpha_r(M^2)-q^2_{2n+1}}\,\frac{1}{\alpha_r(M^2)-(q_1+q_2+\ldots+q_{2n+1})^2}\Big\}=$$
$$=i\,\Big[\alpha_r(M^2)\Big(\frac{\alpha_r(M^2)}{M^2}\Big)^{\beta^2/8\pi}\Big]^2\,\Big[\frac{(-i)}{\alpha_r(M^2)-p^2}\Big]^2$$
$$\times\Big\{\sum^{\infty}_{n=1}\beta^{4n}\int\frac{d^2q_1}{(2\pi)^2i}\,\frac{1}{\alpha_r(M^2)-q^2_1}\int\frac{d^2q_2}{(2\pi)^2i}\,\frac{1}{\alpha_r(M^2)-q^2_2}\ldots$$
$$\times\int\frac{d^2q_{2n}}{(2\pi)^2i}\,\frac{1}{\alpha_r(M^2)-q^2_{2n}}\,\frac{1}{\alpha_r(M^2)-(p-q_1-q_2-\ldots-q_{2n})^2}$$
$$+\sum^{\infty}_{n=0}\beta^{4n+2}\int\frac{d^2q_1}{(2\pi)^2i}\,\frac{1}{\alpha_r(M^2)-q^2_1}\int\frac{d^2q_2}{(2\pi)^2i}\,\frac{1}{\alpha_r(M^2)-q^2_2}\ldots$$
$$\times\int\frac{d^2q_{2n+1}}{(2\pi)^2i}\,\frac{1}{\alpha_r(M^2)-q^2_{2n+1}}\,\frac{1}{\alpha_r(M^2)-(q_1+q_2+\ldots+q_{2n+1})^2}\Big\}. \qquad (3.24)$$

The common factor $[\alpha_r(M^2)Z_1\exp\{\beta^2 i\Delta_F(0,\alpha_r(M^2)/2)\}]^2$ is caused by the summation of the infinite series of one–vertex–loop diagrams. Using (1.4) and (3.19) one can show that it is equal to $\alpha^2_{\rm ph}$:

$$\Big[\alpha_r(M^2)Z_1\exp\Big\{\frac{1}{2}\,\beta^2\,i\Delta_F(0,\alpha_r(M^2))\Big\}\Big]^2=\Big[\alpha_r(M^2)\Big(\frac{\alpha_r(M^2)}{M^2}\Big)^{\beta^2/8\pi}\Big]^2=\alpha^2_{\rm ph}.$$

Thus, we have shown that after renormalization the correction to the two–point Green function to second order in $\alpha_r(M^2)$ and to all orders in the β^2 is proportional to $\alpha^2_{\rm ph}$. Then, since the two–point Green function, calculated to first in $\alpha_r(M^2)$ and to all orders in β^2, is given by (3.18), in the correction of the second order $\alpha_r(M^2)$ we can replace the two–point Green functions of the free sine–Gordon fields in the r.h.s. of (3.24) by (3.18).

This gives

$$
\begin{aligned}
&-i\,\tilde{\Delta}^{(2)}(p,\alpha_r(M^2)) = i\,\alpha^2_{\rm ph}\Big[\frac{(-i)}{\alpha_{\rm ph}-p^2}\Big]^2\Big\{\sum^{\infty}_{n=1}\beta^{4n}\int\frac{d^2q_1}{(2\pi)^2i}\,\frac{1}{\alpha_{\rm ph}-q^2_1}\int\frac{d^2q_2}{(2\pi)^2i}\,\frac{1}{\alpha_{\rm ph}-q^2_2}\ldots\\
&\times\int\frac{d^2q_{2n}}{(2\pi)^2i}\,\frac{1}{\alpha_{\rm ph}-q^2_{2n}}\frac{1}{\alpha_{\rm ph}-(p-q_1-q_2-\ldots-q_{2n})^2}\\
&+\sum^{\infty}_{n=0}\beta^{4n+2}\int\frac{d^2q_1}{(2\pi)^2i}\,\frac{1}{\alpha_{\rm ph}-q^2_1}\int\frac{d^2q_2}{(2\pi)^2i}\,\frac{1}{\alpha_{\rm ph}-q^2_2}\ldots\\
&\times\int\frac{d^2q_{2n+1}}{(2\pi)^2i}\,\frac{1}{\alpha_{\rm ph}-q^2_{2n+1}}\frac{1}{\alpha_{\rm ph}-(q_1+q_2+\ldots+q_{2n+1})^2}\Big\}. \qquad (3.25)
\end{aligned}
$$

In the momentum representation this proves relation (3.22) to second order in $\alpha_r(M^2)$ and to all orders in β^2:

$$
-\,i\,\tilde{\Delta}^{(2)}(p,\alpha_r(M^2)) = -i\,\tilde{\Delta}^{(2)}(p,\alpha_{\rm ph}). \qquad (3.26)
$$

The proof of relation (3.22) to arbitrary orders in $\alpha_r(M^2)$ and β^2 demands only patience and perseverance.

3.3 Non–trivial finite corrections to the dimensional coupling constant $\alpha_{\rm ph}$

As has been shown above quantum fluctuations around the trivial vacuum, calculated to first order in $\alpha_r(M^2)$ and to all orders in β^2, lead to the renormalization of the dimensional coupling constant $\alpha_0(\Lambda^2)$, which reduces to the replacement $\alpha_0(\Lambda^2)\to\alpha_{\rm ph}$, where $\alpha_{\rm ph}$ is the physical (observable) dimensional coupling constant independent on both the cut–off Λ and the normalization scale M. In turn, quantum fluctuations around the trivial vacuum, calculated to second order in $\alpha_r(M^2)$ and to all orders in β^2, induce non–trivial perturbative finite corrections to the physical coupling constant $\alpha_{\rm ph}$.

The simplest correction of this kind is of order $O(\alpha^2_{\rm ph}\beta^2)$. It leads to the perturbative finite correction to the soliton mass, coinciding with that calculated by Dashen *et al.* [6, 7].

Keeping only the terms of order $O(\beta^2)$ in (3.25) we get

$$
\begin{aligned}
-\,i\,\tilde{\Delta}^{(2,1)}(p,\alpha_r(M^2)) &= i\,\alpha^2_{\rm ph}\Big[\frac{(-i)}{\alpha_{\rm ph}-p^2-i\,0}\Big]^2\beta^2\int\frac{d^2q_1}{(2\pi)^2i}\,\frac{1}{(\alpha_{\rm ph}-q^2_1-i\,0)^2} =\\
&= i\,\alpha_{\rm ph}\,\frac{\beta^2}{4\pi}\Big[\frac{(-i)}{\alpha_{\rm ph}-p^2-i\,0}\Big]^2, \qquad (3.27)
\end{aligned}
$$

where we have taken into account that the integral over q_1 is equal to $1/(4\pi\alpha_{\rm ph})$.

The two–point Green function calculated to second order in $\alpha_{\rm ph}$ and to first order in β^2 is equal to

$$
-\,i\,\tilde{\Delta}(p) = \frac{(-i)}{\alpha_{\rm ph}-p^2-i\,0} + i\,\alpha_{\rm ph}\,\frac{\beta^2}{4\pi}\Big[\frac{(-i)}{\alpha_{\rm ph}-p^2-i\,0}\Big]^2 = \frac{(-i)}{\alpha_{\rm eff}-p^2-i\,0}, \qquad (3.28)
$$

where $\alpha_{\rm eff}$ is defined by

$$\alpha_{\rm eff} = \alpha_{\rm ph}\Big(1 - \frac{\beta^2}{4\pi}\Big). \tag{3.29}$$

The effective Lagrangian of the sine–Gordon model defining the two–point Green function (3.28) to leading order in β^2 takes the form

$$\mathcal{L}(x) = \frac{1}{2}\,\partial_{\mu}\vartheta(x)\partial^{\mu}\vartheta(x) + \frac{\alpha_{\rm eff}}{\beta^2}\,(\cos\beta\vartheta(x) - 1). \tag{3.30}$$

The soliton mass, calculated for the effective Lagrangian (3.30), is equal to [6, 7](see also [2])

$$M_s = \frac{8\sqrt{\alpha_{\rm eff}}}{\beta^2} = \frac{8\sqrt{\alpha_{\rm ph}}}{\beta^2} - \frac{\sqrt{\alpha_{\rm ph}}}{\pi}. \tag{3.31}$$

This result coincides with that obtained by Dashen *et al.* [6, 7].

We would like to remind that the finite correction $-\sqrt{\alpha_{\rm ph}}/\pi$ has been interpreted in the literature as a singularity of the sine–Gordon model at $\beta^2 = 8\pi$ (see also [9]). However, as has been conjectured by Zamolodchikov and Zamolodchikov [8], such a singularity of the sine–Gordon model is superficial and depends on the regularization and renormalization procedure. This conjecture has been corroborated in [2].

In our present analysis of the sine–Gordon model the finite correction $-\sqrt{\alpha_{\rm ph}}/\pi$ is a perturbative one. It is valid only for $\beta^2 \ll 8\pi$ and introduces no singularity to the sine–Gordon model at $\beta^2 = 8\pi$.

3.4 Non–trivial momentum dependent corrections to the two–point Green function

Quantum fluctuations around the trivial vacuum, calculated to second order in $\alpha_{\rm ph}$ and to second order in β^2 inclusively, lead to the non–trivial momentum dependence of the two–point Green function of the sine–Gordon field. From (3.25) we find the correction to the two–point Green function to order $O(\alpha^2_{\rm ph}\beta^4)$ inclusively. This reads

$$\begin{aligned} -i\,\tilde{\Delta}^{(2,2)}(p,\alpha_{\rm ph}) &= i\,\alpha^2_{\rm ph}\Big[\frac{(-i)}{\alpha_{\rm ph} - p^2}\Big]^2\Big\{\beta^2\int\frac{d^2q_1}{(2\pi)^2 i}\,\frac{1}{(\alpha_{\rm ph} - q^2_1)^2}\\ &+\beta^4\int\frac{d^2q_1}{(2\pi)^2 i}\,\frac{1}{\alpha_{\rm ph} - q^2_1}\int\frac{d^2q_2}{(2\pi)^2 i}\,\frac{1}{\alpha_{\rm ph} - q^2_2}\,\frac{1}{\alpha_{\rm ph} - (p - q_1 - q_2)^2}\Big\}. \end{aligned} \tag{3.32}$$

The momentum integral of the contribution of order $O(\beta^2)$ is equal to $1/(4\pi\alpha_{\rm ph})$. For the calculation of the momentum integrals of the term of order $O(\beta^4)$ we apply the Feynman parameterization technique. This gives

$$\begin{aligned} &\int\frac{d^2q_1}{(2\pi)^2 i}\,\frac{1}{\alpha_{\rm ph} - q^2_1}\int\frac{d^2q_2}{(2\pi)^2 i}\,\frac{1}{\alpha_{\rm ph} - q^2_2}\,\frac{1}{\alpha_{\rm ph} - (p - q_1 - q_2)^2} =\\ &= \frac{1}{16\pi^2}\int^1_0\!\int^1_0\!\int^1_0\frac{d\eta_1 d\eta_2 d\eta_3\,\delta(1 - \eta_1 - \eta_2 - \eta_3)}{\alpha_{\rm ph}(\eta_1\eta_2 + \eta_2\eta_3 + \eta_3\eta_1) + (-p^2\,)\eta_1\eta_2\eta_3} = \end{aligned}$$

$$= \frac{1}{16\pi^2}\int_0^1 d\eta \int_0^1 d\xi\, \frac{1}{\alpha_{\rm ph}\eta + (\alpha_{\rm ph} - p^2\,\eta)\,(1-\eta)\,\xi\,(1-\xi)} =$$
$$= -\frac{1}{8\pi^2}\int_0^1 \frac{d\eta}{\sqrt{(\alpha_{\rm ph} - p^2\eta)(1-\eta)(4\alpha_{\rm ph}\eta + (\alpha_{\rm ph} - p^2\eta)(1-\eta))}}$$
$$\times\, \ell n\Big(\frac{\sqrt{(\alpha_{\rm ph} - p^2\eta)(1-\eta)} + \sqrt{4\alpha_{\rm ph}\eta + (\alpha_{\rm ph} - p^2\eta)(1-\eta)}}{\sqrt{(\alpha_{\rm ph} - p^2\eta)(1-\eta)} - \sqrt{4\alpha_{\rm ph}\eta + (\alpha_{\rm ph} - p^2\eta)(1-\eta)}}\Big). \tag{3.33}$$

We propose to analyse the behaviour of this integral in the asymptotic regime $p^2 \to \infty$, where it can be calculated analytically. In this limit the main contribution to the integral over η comes from the domain $\eta \sim \alpha_{\rm ph}/p^2$. Therefore, the integrand can be transcribed into the form

$$\int \frac{d^2q_1}{(2\pi)^2 i}\,\frac{1}{\alpha_{\rm ph} - q_1^2}\int \frac{d^2q_2}{(2\pi)^2 i}\,\frac{1}{\alpha_{\rm ph} - q_2^2}\,\frac{1}{\alpha_{\rm ph} - (p - q_1 - q_2)^2} =$$
$$= -\frac{1}{8\pi^2}\int_0^1 \frac{d\eta}{(\alpha_{\rm ph} - p^2\eta)}\,\ell n\Big(\frac{p^2\eta - \alpha_{\rm ph}}{\alpha_{\rm ph}\eta}\Big) = \frac{1}{16\pi^2}\,\frac{1}{p^2}\,\ell n^2\Big(\frac{p^2}{\alpha_{\rm ph}}\Big) + \ldots. \tag{3.34}$$

The correction to the two–point Green function of order $O(\alpha^2_{\rm ph}\beta^4)$ inclusively, taken in the asymptotic regime $p^2 \to \infty$, is equal to

$$-\,i\,\tilde{\Delta}^{(2,2)}(p,\alpha_{\rm ph}) = i\,\alpha_{\rm ph}\Big[\frac{(-i)}{\alpha_{\rm ph} - p^2}\Big]^2\,\frac{\beta^2}{4\pi}\,\Big[1 + \frac{\beta^2}{4\pi}\,\frac{\alpha_{\rm ph}}{p^2}\,\ell n^2\Big(\frac{p^2}{\alpha_{\rm ph}}\Big) + \ldots\Big]. \tag{3.35}$$

Hence, the two–point Green function of the sine–Gordon field, accounting for the contributions of order $O(\alpha^2_{\rm ph}\beta^4)$ inclusively, takes the form

$$\tilde{\Delta}^{-1}(p) = \alpha_{\rm ph}\Big[1 - \frac{\beta^2}{4\pi} - \frac{\beta^4}{16\pi^2}\,\frac{\alpha_{\rm ph}}{p^2}\,\ell n^2\Big(\frac{p^2}{\alpha_{\rm ph}}\Big) + \ldots\Big] - p^2. \tag{3.36}$$

This shows that (i) all divergences can be removed by the renormalization of the dimensional coupling constant $\alpha_0(\Lambda^2)$, (ii) the renormalized expressions are defined in terms of the physical coupling constant $\alpha_{\rm ph}$, (iii) higher order corrections in $\alpha_{\rm ph}$ introduce a non–trivial momentum dependence and (iv) in the asymptotic limit $p^2 \to \infty$ the two–point Green function of the sine–Gordon field behaves as $\tilde{\Delta}(p) \to 1/(-p^2)$. Such a behaviour is confirmed by the analysis of the two–point Green functions with the Callan–Symanzik equation (see Section 4).

3.5 Physical renormalization of the sine–Gordon model

Using the results obtained above we can formulate a procedure for the renormalization of the sine–Gordon model dealing with physical parameters only. Starting with the Lagrangian (1.1) and making a renormalization at the normalization scale $M^2 = \alpha_{\rm ph}$ we deal with physical parameters only

$$\alpha_{\rm ph} = Z_1^{-1}(\beta^2,\alpha_{\rm ph};\Lambda^2)\,\alpha_0(\Lambda^2), \tag{3.37}$$

where the renormalization constant $Z_1(\beta^2,\alpha_{\rm ph};\Lambda^2)$ is equal to

$$Z_1(\beta^2,\alpha_{\rm ph};\Lambda^2) = \Big(\frac{\Lambda^2}{\alpha_{\rm ph}}\Big)^{\beta^2/8\pi}. \tag{3.38}$$

The renormalized Lagrangian is defined by

$$\mathcal{L}(x) = \frac{1}{2}\,(\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha_{\rm ph}}{\beta^2}\,(\cos\beta\vartheta(x) - 1) + (Z_1 - 1)\,\frac{\alpha_{\rm ph}}{\beta^2}\,(\cos\beta\vartheta(x) - 1) \quad (3.39)$$

with the renormalization constant given by Eq.(3.37). From the relation (3.20) at $M^2 = \alpha_{\rm ph}$ one can obtain that

$$\alpha_r(\alpha_{\rm ph}) = \alpha_{\rm ph}. \quad (3.40)$$

The calculation of perturbative corrections to the two–point Green function of the sine–Gordon model shows that the first order correction in $\alpha_{\rm ph}$ vanishes in accordance with Eq.(3.17). Non–trivial corrections appear only to second and higher orders in $\alpha_{\rm ph}$.

4 Renormalization group analysis

In this Section we discuss the renormalization group approach [3]–[5] to the renormalization of the sine–Gordon model. We apply the Callan–Symanzik equation to the analysis of the Fourier transform of the two–point Green function of the sine–Gordon field.

The Callan–Symanzik equation for the Fourier transform of the two–point Green function of the sine–Gordon field (3.5), which we denote below as $-i\,\tilde{\Delta}(p;\alpha_r(M^2),\beta^2)$, is equal to [3]

$$\Big[-p\cdot\frac{\partial}{\partial p} + \beta(\alpha_r(M^2),\beta^2)\,\frac{\partial}{\partial\alpha_r(M^2)} - 2\Big]\tilde{\Delta}(p;\alpha_r(M^2),\beta^2) = F(0,p;\alpha_r(M^2),\beta^2), \quad (4.1)$$

where $\beta(\alpha_r(M^2),\beta^2)$ is the Gell–Mann–Low function

$$M\,\frac{\partial\alpha_r(M^2)}{\partial M} = \beta(\alpha_r(M^2),\beta^2). \quad (4.2)$$

The term $\gamma(\alpha_r(M^2),\beta^2)$ [3], describing an anomalous dimension of the sine–Gordon field, does not appear in the Callan–Symanzik equation (4.1) due to unrenormalizability of the sine–Gordon field $\vartheta(x)$. The r.h.s. of (4.1) is defined by

$$F(0,p;\alpha_r(M^2),\beta^2) = \iint d^2x d^2y\, e^{+ip\,\cdot\,x}\,\langle 0|{\rm T}\Big(\Theta^\mu_\mu(y)\vartheta(x)\vartheta(0)\,e^{i\int d^2y\,\mathcal{L}_{\rm int}(y)}\Big)|0\rangle_c, \quad (4.3)$$

where $\mathcal{L}_{\rm int}(y)$ is equal to [2]

$$\mathcal{L}_{\rm int}(y) = \frac{\alpha_0}{\beta^2}\,(\cos\beta\vartheta(x) - 1). \quad (4.4)$$

Then, $\Theta^\mu_\mu(y)$ is the trace of the energy–momentum tensor $\Theta_{\mu\nu}(x)$. For a (pseudo)scalar field $\vartheta(x)$, described by the Lagrangian $\mathcal{L}(x)$, it is defined by [3]

$$\Theta_{\mu\nu}(x) = \frac{\partial\mathcal{L}(x)}{\partial^\mu\vartheta(x)}\,\partial_\nu\vartheta(x) - g_{\mu\nu}\,\mathcal{L}(x). \quad (4.5)$$

Using the Lagrange equation of motion one can show that

$$\partial^\mu\Theta_{\mu\nu}(x) = 0. \quad (4.6)$$

For the sine–Gordon model the energy–momentum tensor $\Theta_{\mu\nu}(x)$ reads

$$\Theta_{\mu\nu}(x) = \partial_\mu\vartheta(x)\partial_\nu\vartheta(x) - g_{\mu\nu}\Big[\frac{1}{2}\partial_\lambda\vartheta(x)\partial^\lambda\vartheta(x) + \frac{\alpha_0}{\beta^2}\,(\cos\beta\vartheta(x) - 1)\Big]. \tag{4.7}$$

The trace of the energy–momentum tensor $\Theta_{\mu\nu}(x)$ is equal to

$$\Theta^\mu_\mu(x) = -\,\frac{2\alpha_0}{\beta^2}\,(\cos\beta\vartheta(x) - 1) = 2\,V[\vartheta(x)], \tag{4.8}$$

where $V[\vartheta(x)]$ is the potential density functional of the sine–Gordon field $\vartheta(x)$.

Since the trace of the energy–momentum tensor is proportional to the potential energy density, the Fourier transform $F(0,p;\alpha_r(M^2),\beta^2)$ can be related to the two–point Green function as

$$F(0,p;\alpha_r(M^2),\beta^2) = 2\,\alpha_r(M^2)\,\frac{\partial}{\partial\alpha_r(M^2)}\tilde{\Delta}(p;\alpha_r(M^2),\beta^2), \tag{4.9}$$

where we have used the definition of the trace $\Theta^\mu_\mu(y)$ of the energy–momentum tensor Eq.(4.8) and the relation $\alpha_0 = \alpha_r(M^2)\,Z_1(\beta^2, M^2;\Lambda^2)$.

Substituting (4.9) into (4.1) we arrive at the Callan–Symanzik equation for the Fourier transform of the two–point Green function of the sine–Gordon field

$$\Big[-p^2\frac{\partial}{\partial p^2} + \Big(\frac{1}{2}\,\beta(\alpha_r(M^2),\beta^2) - \alpha_r(M^2)\Big)\,\frac{\partial}{\partial\alpha_r(M^2)} - 1\Big]\tilde{\Delta}(p^2;\alpha_r(M^2),\beta^2) = 0, \tag{4.10}$$

where we have taken into account that $\tilde{\Delta}(p;\alpha_r(M^2),\beta^2)$ should depend on p^2 due to Lorentz covariance.

For the solution of (4.10) we have to determine the Gell–Mann–Low function (4.2). For the coupling constant $\alpha_r(M^2)$, defined by (3.20), the Gell–Mann–Low function is

$$\beta(\alpha_r(M^2),\beta^2) = \frac{\tilde{\beta}^2}{4\pi}\,\alpha_r(M^2), \tag{4.11}$$

where $\tilde{\beta}^2 = \beta^2/(1 + \beta^2/8\pi)$ (3.20). This gives the Callan–Symanzik equation

$$\Big[p^2\frac{\partial}{\partial p^2} + \Big(1 - \frac{\tilde{\beta}^2}{8\pi}\Big)\,\alpha_r(M^2)\,\frac{\partial}{\partial\alpha_r(M^2)} + 1\Big]\tilde{\Delta}(p^2;\alpha_r(M^2),\beta^2) = 0. \tag{4.12}$$

Setting $\tilde{\Delta}(p^2;\alpha_r(M^2),\beta^2) = D(p^2;\alpha_r(M^2),\beta^2)/p^2$ we get

$$\Big[p^2\frac{\partial}{\partial p^2} + \Big(1 - \frac{\tilde{\beta}^2}{8\pi}\Big)\,\alpha_r(M^2)\,\frac{\partial}{\partial\alpha_r(M^2)}\Big]D(p^2;\alpha_r(M^2),\beta^2) = 0. \tag{4.13}$$

Due to dimensional consideration the function $D(p^2;\alpha_r(M^2),\beta^2)$ should be dimensionless, depending on the dimensionless variables $\tilde{p}^2 = p^2/M^2$ and $\tilde{\alpha} = \alpha_r(M^2)/M^2$, where M is a normalization scale. This gives

$$\Big[\tilde{p}^2\frac{\partial}{\partial\tilde{p}^2} + \Big(1 - \frac{\tilde{\beta}^2}{8\pi}\Big)\,\tilde{\alpha}\,\frac{\partial}{\partial\tilde{\alpha}}\Big]D(\tilde{p}^2;\tilde{\alpha},\beta^2) = 0. \tag{4.14}$$

According to the general theory of partial differential equations of first order [10], the solution of (4.14) is an arbitrary function of the integration constant

$$C = \frac{\tilde{\alpha}}{\tilde{p}^2}\,(\tilde{p}^2)^{\tilde{\beta}^2/8\pi}, \tag{4.15}$$

which is the solution of the characteristic differential equation

$$\Big(1 - \frac{\tilde{\beta}^2}{8\pi}\Big)\frac{d\tilde{p}^2}{\tilde{p}^2} = \frac{d\tilde{\alpha}}{\tilde{\alpha}}. \tag{4.16}$$

Hence, the Fourier transform of the two–point Green function of the sine–Gordon field is equal to

$$\tilde{\Delta}(p^2;\alpha_r(M^2),\beta^2) = \frac{1}{p^2}\,D\Big[\frac{\alpha_r(M^2)}{p^2}\,\Big(\frac{p^2}{M^2}\Big)^{\tilde{\beta}^2/8\pi}\Big]. \tag{4.17}$$

The argument of the D–function can be expressed in terms of the running coupling constant $\alpha_r(p^2)$:

$$\alpha_r(p^2) = \alpha_r(M^2)\Big(\frac{p^2}{M^2}\Big)^{\tilde{\beta}^2/8\pi} = \alpha_{\rm ph}\Big(\frac{M^2}{\alpha_{\rm ph}}\Big)^{\tilde{\beta}^2/8\pi}\Big(\frac{p^2}{M^2}\Big)^{\tilde{\beta}^2/8\pi} = \alpha_{\rm ph}\Big(\frac{p^2}{\alpha_{\rm ph}}\Big)^{\tilde{\beta}^2/8\pi}. \tag{4.18}$$

The solution of the Callan–Symanzik equation for the Fourier transform of the two–point Green function of the sine–Gordon field is

$$\tilde{\Delta}(p^2;\alpha_{\rm ph},\beta^2) = \frac{1}{p^2}\,D\Big[\frac{\alpha_r(p^2)}{p^2}\Big]. \tag{4.19}$$

This proves that the total renormalized two–point Green function of the sine–Gordon field depends on the physical coupling constant $\alpha_{\rm ph}$ only.

A perturbative calculation of the two–point Green function, carried out in Section 3, gives the following expression for the function $D[\alpha_r(p^2)/p^2]$ in the asymptotic region $p^2 \to \infty$:

$$D\Big[\frac{\alpha_r(p^2)}{p^2}\Big] = \Big\{\frac{\alpha_{\rm ph}}{p^2}\Big[1 - \frac{\beta^2}{4\pi} - \frac{\beta^4}{16\pi^2}\,\frac{\alpha_{\rm ph}}{p^2}\,\ell n^2\Big(\frac{p^2}{\alpha_{\rm ph}}\Big) + \ldots\Big] - 1\Big\}^{-1}. \tag{4.20}$$

Unfortunately, this is not able to reproduce the non–perturbative expression of the two–point Green function $\tilde{\Delta}(p^2;\alpha_{\rm ph},\beta^2)$.

5 Renormalization of Gaussian fluctuations around solitons

We apply the renormalization procedure expounded above to the calculation of the contribution of quantum fluctuations around a soliton solution. We start with the partition function

$$\begin{aligned} Z_{\rm SG} &= \int \mathcal{D}\vartheta\,\exp\Big\{i\int d^2x\,\Big[\frac{1}{2}\,\partial_{\mu}\vartheta(x)\partial^{\mu}\vartheta(x) + \frac{\alpha_0}{\beta^2}\,(\cos\beta\vartheta(x) - 1)\Big]\Big\} = \\ &= \int \mathcal{D}\vartheta\,\exp\Big\{i\int d^2x\,\mathcal{L}[\vartheta(x)]\Big\}. \end{aligned} \tag{5.1}$$

Following Dashen *et al.* [6, 7] (see also [11]) we treat the quantum fluctuations of the sine–Gordon field $\vartheta(x)$ around the classical solution $\vartheta(x) = \vartheta_{c\ell}(x) + \varphi(x)$, where $\varphi(x)$ is the field fluctuating around $\vartheta_{c\ell}(x)$, the single soliton solution of the classical equation of motion

$$\Box\vartheta_{c\ell}(x) + \frac{\alpha_0}{\beta}\,\sin\beta\vartheta_{c\ell}(x) = 0 \tag{5.2}$$

equal to [6, 9, 11]

$$\vartheta_{c\ell}(x) = \frac{4}{\beta}\,\arctan(\exp(\sqrt{\alpha_0}\,\gamma(x^1 - ux^0)) = \frac{4}{\beta}\,\arctan(\exp(\sqrt{\alpha_0}\,\sigma)), \tag{5.3}$$

where u is the velocity of the soliton, $\sigma = \gamma(x^1 - ux^0)$ and $\gamma = 1/\sqrt{1-u^2}$ [1].

Substituting $\vartheta(x) = \vartheta_{c\ell}(x) + \varphi(x)$ into the exponent of the integrand of (5.1) and using the equation of motion for the soliton solution $\vartheta_{c\ell}(x)$ we get

$$\begin{aligned}Z_{\rm SG} = \exp\Big\{i\int d^2x\,\mathcal{L}[\vartheta_{c\ell}(x)]\Big\}\int\mathcal{D}\varphi\,\exp\Big\{i\int d^2x\,\Big[\frac{1}{2}\,\partial_{\mu}\varphi(x)\partial^{\mu}\varphi(x)\\ +\frac{\alpha_0}{\beta^2}\,\sin\beta\vartheta_{c\ell}(x)\,(\beta\varphi(x) - \sin\beta\varphi(x)) + \frac{\alpha_0}{\beta^2}\,\cos\beta\vartheta_{c\ell}(x)(\cos\beta\varphi(x) - 1)\Big]\Big\}.\end{aligned} \tag{5.4}$$

Substituting (5.3) into (5.4) we obtain

$$\begin{aligned}\mathcal{L}[\varphi(x)] &= \frac{1}{2}\,\partial_{\mu}\varphi(x)\partial^{\mu}\varphi(x) + \frac{\alpha_0}{\beta^2}\,(\cos\beta\varphi(x) - 1)\\ &- \frac{2\alpha_0}{\beta^2}\,\frac{1}{\cosh^2(\sqrt{\alpha_0}\,\gamma(x^1 - ux^0))}(\cos\beta\varphi(x) - 1)\\ &- \frac{2\alpha_0}{\beta^2}\,\frac{\sinh(\sqrt{\alpha_0}\,\gamma(x^1 - ux^0))}{\cosh^2(\sqrt{\alpha_0}\,\gamma(x^1 - ux^0))}\,(\beta\varphi(x) - \sin\beta\varphi(x)).\end{aligned} \tag{5.5}$$

In terms of σ and τ the exponent of the partition function (5.4) reads

$$\begin{aligned}&Z_{\rm SG} = \exp\Big\{i\int d^2x\,\mathcal{L}[\vartheta_{c\ell}(x)]\Big\}\\ &\int\mathcal{D}\varphi\,\exp\Big\{i\int d\tau d\sigma\,\Big[\frac{1}{2}\,\partial_{\mu}\varphi(\tau,\sigma)\partial^{\mu}\varphi(\tau,\sigma) + \frac{\alpha_0}{\beta^2}\,(\cos\beta\varphi(\tau,\sigma) - 1)\\ &-\frac{2\alpha_0}{\beta^2}\,\frac{1}{\cosh^2(\sqrt{\alpha_0}\sigma)}(\cos\beta\varphi(x) - 1) - \frac{2\alpha_0}{\beta^2}\,\frac{\sinh(\sqrt{\alpha_0}\sigma)}{\cosh^2(\sqrt{\alpha_0}\sigma)}\,(\beta\varphi(\tau,\sigma) - \sin\beta\varphi(\tau,\sigma))\Big]\Big\},\end{aligned} \tag{5.6}$$

where we have denoted

$$\partial_{\mu}\varphi(\tau,\sigma)\partial^{\mu}\varphi(\tau,\sigma) = \Big(\frac{\partial\varphi(\tau,\sigma)}{\partial\tau}\Big)^2 - \Big(\frac{\partial\varphi(\tau,\sigma)}{\partial\sigma}\Big)^2. \tag{5.7}$$

[1] In analogy with the "spatial" variable σ we can define the "time" variable for the soliton moving with velocity u as $\tau = \gamma(x^0 - ux^1)$. In variables (τ,σ) an infinitesimal element of the 2–dimensional volume d^2x is equal to $d^2x = d\tau d\sigma$ and the d'Alembert operator $\Box$ is defined by $\Box = \partial^2/\partial\tau^2 - \partial^2/\partial\sigma^2$.

The equation of motion for the fluctuating field $\varphi(\tau,\sigma)$ is equal to

$$\begin{aligned}\Box\varphi(\tau,\sigma) + \frac{\alpha_0}{\beta}\sin\beta\varphi(\tau,\sigma) &= +2\,\frac{\alpha_0}{\beta}\,\frac{1}{\cosh^2(\sqrt{\alpha_0}\sigma)}\,\sin\beta\varphi(\tau,\sigma)\\ &\quad -2\,\frac{\alpha_0}{\beta}\,\frac{\sinh(\sqrt{\alpha_0}\sigma)}{\cosh^2(\sqrt{\alpha_0}\sigma)}\,(1-\cos\beta\varphi(\tau,\sigma)). \qquad (5.8)\end{aligned}$$

Dealing with Gaussian fluctuations only [6, 7] and keeping the squared terms in the Lagrangian (5.5) in the fluctuating field $\varphi(x)$ expansion only, we transcribe the partition function (5.6) into the form

$$\begin{aligned}&Z_{\rm SG} = \exp\Big\{i\int d^2x\,\mathcal{L}[\vartheta_{c\ell}(x)]\Big\}\\ &\times\int\mathcal{D}\varphi\,\exp\Big\{-i\,\frac{1}{2}\int d\tau d\sigma\,\varphi(\tau,\sigma)\Big[\Box + \alpha_0 - \frac{2\alpha_0}{\cosh^2(\sqrt{\alpha_0}\sigma)}\Big]\varphi(\tau,\sigma)\Big\}. \qquad (5.9)\end{aligned}$$

It is seen that $\sqrt{\alpha_0}$ has the distinct meaning of the mass of the quanta of the Klein–Gordon field $\varphi(\tau,\sigma)$ coupled to an external force described by a scalar potential [2].

Integrating over the fluctuating field $\varphi(\tau,\sigma)$ we transcribe the r.h.s. of (5.9) into the form

$$Z_{\rm SG} = \exp\Big\{i\int d^2x\,\mathcal{L}[\vartheta_{c\ell}(x)] + i\delta\mathcal{S}[\vartheta_{c\ell}]\Big\}. \qquad (5.10)$$

We have denoted

$$\begin{aligned}\exp\{i\delta\mathcal{S}[\vartheta_{c\ell}]\} &= \exp\{i\int d^2x\,\delta\mathcal{L}_{\rm eff}[\vartheta_{c\ell}(x)]\} = \sqrt{\frac{{\rm Det}(\Box+\alpha_0)}{{\rm Det}\Big(\Box+\alpha_0-\dfrac{2\alpha_0}{\cosh^2(\sqrt{\alpha_0}\sigma)}\Big)}} =\\ &= \exp\Big\{-\frac{1}{2}\sum_n \ell n\,\lambda_n + \frac{1}{2}\int d^2x\int\frac{d^2p}{(2\pi)^2}\,\ell n(\alpha_0 - p^2)\Big\}, \qquad (5.11)\end{aligned}$$

where p is a 1+1–dimensional momentum. The second term in the exponent corresponds to the subtraction of the vacuum contribution. The effective action, caused by fluctuations around a soliton solution, is defined by

$$\delta\mathcal{S}[\vartheta] = \int d^2x\,\delta\mathcal{L}_{\rm eff}[\vartheta_{c\ell}(x)] = i\,\frac{1}{2}\sum_n \ell n\,\lambda_n + \frac{1}{2}\int d^2x\int\frac{d^2p}{(2\pi)^2 i}\,\ell n(\alpha_0 - p^2), \qquad (5.12)$$

where λ_n are the eigenvalues of the equation

$$\Big(\Box + \alpha_0 - \frac{2\alpha_0}{\cosh^2(\sqrt{\alpha_0}\sigma)}\Big)\,\varphi_n(\tau,\sigma) = \lambda_n\,\varphi_n(\tau,\sigma) \qquad (5.13)$$

and $\varphi_n(\tau,\sigma)$ are eigenfunctions. The quantum number n can be both discrete and continuous. This implies that the product over n in (5.11) should contain both the summation

[2]The parameter α_0 should enter with the imaginary correction $\alpha_0 \to \alpha_0 - i\,0$. This is required by the convergence of the path integral [13].

over the discrete values of the quantum number n and integration over the continuous ones.

According to the Fourier method [12], the solution of equation (5.13) should be taken in the form

$$\varphi_n(\tau,\sigma) = e^{-i\,\omega\tau}\,\psi_n(\sigma), \tag{5.14}$$

where $-\infty \le \omega \le +\infty$ and $\psi_n(\sigma)$ is a complex function[3].

Substituting (5.14) into (5.13) we get

$$\Big(\frac{d^2}{d\sigma^2} + k^2 + \frac{2\alpha_0}{\cosh^2(\sqrt{\alpha_0}\,\sigma)}\Big)\,\psi_n(\tau,\sigma) = 0, \tag{5.15}$$

where we have denoted

$$k^2 = \lambda_n + \omega^2 - \alpha_0. \tag{5.16}$$

This defines eigenvalues λ_n as functions of ω and k

$$\lambda_n = \alpha_0 - \omega^2 + k^2. \tag{5.17}$$

The parameter k has the meaning of a spatial momentum $-\infty < k < +\infty$. The solutions of equation (5.15) are[4]

$$\begin{aligned}\psi_b(\sigma) &= \sqrt{\frac{\sqrt{\alpha_0}}{2}}\,\frac{1}{\cosh(\sqrt{\alpha_0}\sigma)},\\ \psi_k(\sigma) &= \frac{i}{\sqrt{2\pi}}\,\frac{-ik + \sqrt{\alpha_0}\,\tanh(\sqrt{\alpha_0}\sigma)}{\sqrt{k^2+\alpha_0}}\,e^{+ik\sigma},\end{aligned} \tag{5.18}$$

where the eigenfunction $\psi_b(\sigma)$ has eigenvalues $\lambda_n = -\omega^2$ and the eigenfunctions $\psi_k(\sigma)$ have eigenvalues $\lambda_n = \alpha_0 - \omega^2 + k^2$. In the asymptotic region $\sigma \to \infty$ the function $\psi_k(\sigma)$ behaves as

$$\psi_k(\sigma) \to \frac{1}{\sqrt{2\pi}}\,e^{+ik\sigma + i\,\frac{1}{2}\,\delta(k)}, \tag{5.19}$$

where $\delta(k)$ is a phase shift defined by [11]

$$\delta(k) = 2\arctan\frac{\sqrt{\alpha_0}}{k}. \tag{5.20}$$

The solutions (5.18) satisfy the completeness condition [11] (see Appendix)

$$\int^{+\infty}_{-\infty} dk\,\psi^*_k(\sigma\,')\psi_k(\sigma) + \psi_b(\sigma\,')\psi_b(\sigma) = \delta(\sigma\,' - \sigma). \tag{5.21}$$

[3] Since $\varphi_n(\tau,\sigma)$ is a real field, we have to take the real part of the solution (5.14) only, i.e. $\varphi(\tau,\sigma) = \mathcal{R}e\,(e^{-i\,\omega\tau}\,\psi(\sigma))$. Though without loss of generality one can also use complex eigenfunctions [7, 9, 11].

[4] The solutions of the equation (5.15) are well–known [11] (see also [7, 9]). Nevertheless, we adduce the solution of this equation in the Appendix.

The fluctuating field $\varphi(\tau,\sigma)$ is equal to (see (A.12))

$$\begin{aligned}
\varphi_{\omega b}(\tau,\sigma) &= \frac{1}{\sqrt{2\pi}}\,e^{-i\omega\tau}\psi_b(\sigma) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\sqrt{\alpha_0}}{2}}\,\frac{1}{\cosh(\sqrt{\alpha_0}\sigma)}\,e^{-i\omega\tau},\\
\varphi_{\omega k}(\tau,\sigma) &= \frac{1}{\sqrt{2\pi}}\,e^{-i\omega\tau}\psi_k(\sigma) = \frac{i}{2\pi}\,\frac{-ik+\sqrt{\alpha_0}\,\tanh(\sqrt{\alpha_0}\sigma)}{\sqrt{k^2+\alpha_0}}\,e^{-i\omega\tau+ik\sigma}.
\end{aligned} \tag{5.22}$$

In terms of the eigenvalues $\lambda_n = -\,\omega^2$ and $\lambda_n = \alpha_0 - \omega^2 + k^2$ and eigenfunctions (5.22) the effective action $\delta\mathcal{S}[\vartheta_{\mathrm{c}\ell}]$ is determined by

$$\begin{aligned}
&\delta\mathcal{S}[\vartheta_{\mathrm{c}\ell}] = -\frac{1}{2}\int d^2x\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi i}\int_{-\infty}^{+\infty}dk\,|\psi_k(x)|^2\,\ell n(\alpha_0-\omega^2+k^2)\\
&-\frac{1}{2}\int d^2x\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi i}\,|\psi_b(x)|^2\,\ell n(-\omega^2)+\frac{1}{2}\int d^2x\int\frac{d^2p}{(2\pi)^2 i}\,\ell n(\alpha_0-p^2).
\end{aligned} \tag{5.23}$$

Using the explicit expressions for the eigenfunctions $\psi_k(x)$ and $\psi_b(x)$ we reduce the r.h.s. of (5.23) to the form

$$\begin{aligned}
&\delta\mathcal{S}[\vartheta_{\mathrm{c}\ell}] = -\frac{1}{2}\int d^2x\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi i}\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\,\frac{k^2+\alpha_0\tanh^2(\sqrt{\alpha_0}\sigma)}{k^2+\alpha_0}\,\ell n(\alpha_0-\omega^2+k^2)\\
&-\frac{1}{2}\int d^2x\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi i}\,\frac{\sqrt{\alpha_0}}{2}\,\frac{1}{\cosh^2(\sqrt{\alpha_0}\sigma)}\,\ell n(-\omega^2)+\frac{1}{2}\int d^2x\int\frac{d^2p}{(2\pi)^2 i}\,\ell n(\alpha_0-p^2).
\end{aligned} \tag{5.24}$$

Introducing the notation

$$\frac{1}{\cosh^2(\sqrt{\alpha_0}\sigma)} = \frac{1}{2}\,(1-\cos\beta\vartheta_{\mathrm{c}\ell}(x)) = \frac{\beta^2}{2\alpha_0}\,V[\vartheta_{\mathrm{c}\ell}(x)] \tag{5.25}$$

we obtain the effective Lagrangian $\delta\mathcal{L}_{\mathrm{eff}}[\vartheta_{\mathrm{c}\ell}(x)]$. It is equal to [5]

$$\begin{aligned}
\delta\mathcal{L}_{\mathrm{eff}}[\vartheta_{\mathrm{c}\ell}(x)] &= -\frac{1}{4}\beta^2 V[\vartheta_{\mathrm{c}\ell}(x)]\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi i}\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\,\frac{1}{k^2+\alpha_0}[\ell n(-\omega^2)-\ell n(\alpha_0-\omega^2+k^2)]\\
&- \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi i}\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\ell n(\alpha_0-\omega^2+k^2)+\frac{1}{2}\int\frac{d^2p}{(2\pi)^2 i}\ell n(\alpha_0-p^2).
\end{aligned} \tag{5.26}$$

The two last terms cancel each other. This gives

$$\delta\mathcal{L}_{\mathrm{eff}}[\vartheta_{\mathrm{c}\ell}(x)] = -\frac{1}{4}\beta^2 V[\vartheta_{\mathrm{c}\ell}(x)]\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi i}\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\,\frac{1}{k^2+\alpha_0}[\ell n(-\omega^2)-\ell n(\alpha_0-\omega^2+k^2)]. \tag{5.27}$$

After the integration by parts over ω the effective Lagrangian $\delta\mathcal{L}_{\mathrm{eff}}[\vartheta_{\mathrm{c}\ell}(x)]$ is defined by

$$\delta\mathcal{L}_{\mathrm{eff}}[\vartheta_{\mathrm{c}\ell}(x)] = \frac{1}{2}\,\beta^2 V[\vartheta_{\mathrm{c}\ell}(x)]\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi i}\,\frac{1}{\alpha_0-\omega^2+k^2-i\,0}. \tag{5.28}$$

[5] In the contribution of the mode $\lambda_n = -\omega^2$ we have used the integral representation

$$\frac{1}{2\sqrt{\alpha_0}} = \int_{-\infty}^{+\infty}\frac{dk}{2\pi}\,\frac{1}{k^2+\alpha_0}.$$

The appearance of the imaginary correction $-i\,0$ is caused by the convergence of the path integral (5.9) [13].

Integrating over ω we reduce the r.h.s. of (5.28) to the form

$$\begin{aligned}\delta\mathcal{L}_{\rm eff}[\vartheta_{\rm c\ell}(x)] &= \frac{1}{2}\,\beta^2 V[\vartheta_{\rm c\ell}(x)]\int^{+\infty}_{-\infty}\frac{dk}{4\pi}\frac{1}{\sqrt{\alpha_0+k^2}} = \\ &= -\,\frac{\beta^2}{4\sqrt{\alpha_0}}\,V[\vartheta_{\rm c\ell}(x)]\int^{+\infty}_{-\infty}\frac{dk}{4\pi}\,\sqrt{\alpha_0+k^2}\,\frac{d\delta(k)}{dk},\end{aligned}\tag{5.29}$$

where $\delta(k)$ is a phase shift defined in Eq.(5.20). We discuss this expression in Section 7 in connection with the correction to the soliton mass caused by Gaussian fluctuations.

For the analysis of the renormalizability of the sine–Gordon model the momentum integral in the effective Lagrangian $\delta\mathcal{L}_{\rm eff}[\vartheta_{\rm c\ell}(x)]$ should be taken in a Lorentz covariant form (5.28) and regularized in a covariant way. Making a Wick rotation $\omega \to i\,\omega$ and passing to Euclidean momentum space we define the integral over ω and k in (5.28) as [2]

$$\int^{+\infty}_{-\infty}\frac{dk}{2\pi}\int^{+\infty}_{-\infty}\frac{d\omega}{2\pi i}\,\frac{1}{\alpha_0-\omega^2+k^2-i\,0} = \frac{1}{4\pi}\,\ell n\Big(\frac{\Lambda^2}{\alpha_0}\Big),\tag{5.30}$$

where Λ is an Euclidean cut–off [2]. The effective Lagrangian, induced by Gaussian fluctuations around a soliton solution, is equal to

$$\delta\mathcal{L}_{\rm eff}[\vartheta_{\rm c\ell}(x)] = \frac{\alpha_0}{\beta^2}\Big[-\frac{\beta^2}{8\pi}\,\ell n\Big(\frac{\Lambda^2}{\alpha_0}\Big)\Big]\,(\cos\beta\vartheta_{\rm c\ell}(x)-1).\tag{5.31}$$

The Lagrangian (5.31) has the distinct form of the correction, caused by quantum fluctuations around the trivial vacuum calculated to first orders in $\alpha_0(\Lambda^2)$ and β^2.

The total Lagrangian, accounting for Gaussian fluctuations around the soliton solution amounts to

$$\mathcal{L}_{\rm eff}(x) = \frac{1}{2}\partial_{\mu}\vartheta_{\rm c\ell}(x)\partial^{\mu}\vartheta_{\rm c\ell}(x) + \frac{\alpha_0}{\beta^2}\Big[1-\frac{\beta^2}{8\pi}\,\ell n\Big(\frac{\Lambda^2}{\alpha_0}\Big)\Big]\,(\cos\beta\vartheta_{\rm c\ell}(x)-1).\tag{5.32}$$

This coincides with Eq.(6.7) of Ref.[2].

As has been shown in [2], the dependence of the effective Lagrangian (5.32) on the cut–off Λ can be removed by renormalization with the renormalization constant (1.4)

$$\begin{aligned}&\alpha_0\Big[1-\frac{\beta^2}{8\pi}\,\ell n\Big(\frac{\Lambda^2}{\alpha_0}\Big)\Big] = \alpha_r(M^2)Z_1\Big[1-\frac{\beta^2}{8\pi}\ell n\Big(\frac{\Lambda^2}{\alpha_r(M^2)Z_1}\Big)\Big] = \\ &= \alpha_r(M^2)\Big[1+\frac{\beta^2}{8\pi}\ell n\Big(\frac{\Lambda^2}{M^2}\Big)-\frac{\beta^2}{8\pi}\ell n\Big(\frac{\Lambda^2}{\alpha_r(M^2)}\Big)\Big] = \alpha_r(M^2)\Big[1+\frac{\beta^2}{8\pi}\ell n\Big(\frac{\alpha_r(M^2)}{M^2}\Big)\Big],\end{aligned}\tag{5.33}$$

where we have kept terms of order $O(\beta^2)$ in the β^2–expansion of the renormalization constant (1.4). This gives the effective Lagrangian

$$\mathcal{L}_{\rm eff}(x) = \frac{1}{2}\partial_{\mu}\vartheta_{\rm c\ell}(x)\partial^{\mu}\vartheta_{\rm c\ell}(x) + \frac{\alpha_r(M^2)}{\beta^2}\Big[1+\frac{\beta^2}{8\pi}\,\ell n\Big(\frac{\alpha_r(M^2)}{M^2}\Big)\Big]\,(\cos\beta\vartheta_{\rm c\ell}(x)-1).\tag{5.34}$$

We can replace the coupling constant $\alpha_r(M^2)$ by the physical coupling constant $\alpha_{\rm ph}$, related to $\alpha_r(M^2)$ by (3.19)

$$\alpha_{\rm ph} = \alpha_r(M^2)\Big[1 + \frac{\beta^2}{8\pi}\,\ell n\Big(\frac{\alpha_r(M^2)}{M^2}\Big)\Big], \tag{5.35}$$

where we have kept terms of order $O(\beta^2)$ only. Substituting (5.35) in (5.34) we get

$$\mathcal{L}^{(r)}_{\rm eff}(x) = \frac{1}{2}\partial_{\mu}\vartheta_{c\ell}(x)\partial^{\mu}\vartheta_{c\ell}(x) + \frac{\alpha_{\rm ph}}{\beta^2}\,(\cos\beta\vartheta_{c\ell}(x) - 1). \tag{5.36}$$

The effective Lagrangian (5.36) coincides with the Lagrangian, renormalized by the quantum fluctuations around the trivial vacuum (3.21), and corroborates the result obtained in [2].

We would like to emphasize that analysing the renormalization of the sine–Gordon model, caused by Gaussian fluctuations around a soliton, one can see that Gaussian fluctuations are perturbative fluctuations of order $O(\alpha_r(M^2)\beta^2)$ valid for $\beta^2 \ll 8\pi$, which cannot be responsible for non–perturbative contributions to the soliton mass at $\beta^2 = 8\pi$.

6 Renormalization of the soliton mass by Gaussian fluctuations in continuous space–time

Using the effective Lagrangian Eq.(5.29) one can calculate the soliton mass corrected by quantum fluctuations. It reads

$$M_s = \frac{8\sqrt{\alpha_0}}{\beta^2} + \Delta M_s, \tag{6.1}$$

where ΔM_s is defined by

$$\Delta M_s = -\int^{+\infty}_{-\infty} dx^1\,\delta\mathcal{L}_{\rm eff}[\vartheta_{c\ell}(x^1)] = \int^{+\infty}_{-\infty}\frac{dk}{4\pi}\,\sqrt{\alpha_0 + k^2}\,\frac{d\delta(k)}{dk}. \tag{6.2}$$

This corresponds to the correction to the soliton mass, induced by Gaussian fluctuations, without a *surface term* $-\sqrt{\alpha_0}/\pi$ [14, 15].

In the Lorentz covariant form the correction to the soliton mass reads

$$\begin{aligned}\Delta M_s &= -\int^{+\infty}_{-\infty} dx^1\,\delta\mathcal{L}_{\rm eff}[\vartheta_{c\ell}(x^1)] = \\ &= \int^{+\infty}_{-\infty}\frac{d\omega}{2\pi i}\int^{+\infty}_{-\infty}\frac{dk}{2\pi}\frac{\sqrt{\alpha_0}}{k^2 + \alpha_0}[\ell n(-\omega^2) - \ell n(\alpha_0 - \omega^2 + k^2)] \\ &= -2\sqrt{\alpha_0}\int^{+\infty}_{-\infty}\frac{dk}{2\pi}\int^{+\infty}_{-\infty}\frac{d\omega}{2\pi i}\frac{1}{\alpha_0 - \omega^2 + k^2 - i\,0},\end{aligned} \tag{6.3}$$

where we have taken the effective Lagrangian defined by (5.27) and integrated over ω by parts. Using the result of the calculation of the integral (5.30) we get the following expression for the soliton mass corrected by Gaussian fluctuations

$$M_s = \frac{8\sqrt{\alpha_0(\Lambda^2)}}{\beta^2} - \frac{\sqrt{\alpha_0(\Lambda^2)}}{2\pi}\,\ell n\Big[\frac{\Lambda^2}{\alpha_0(\Lambda^2)}\Big]. \tag{6.4}$$

The removal of the dependence on the cut–off Λ should be carried out within the renormalization procedure.

Replacing $\alpha_0(\Lambda^2)$ in (6.4) by $\alpha_r(M^2)Z_1(\beta^2, M^2; \Lambda^2)$, where the renormalization constant $Z_1(\beta^2, M^2; \Lambda^2)$ is defined by Eq.(1.4), we get

$$M_s = \frac{8\sqrt{\alpha_r(M^2)Z_1}}{\beta^2} - \frac{\sqrt{\alpha_r(M^2)\,Z_1}}{2\pi}\,\ell n\Big[\frac{\Lambda^2}{\alpha_r(M^2)Z_1}\Big]. \tag{6.5}$$

The renormalization constant Z_1 should be expanded in power of β^2 to order $O(\beta^2)$. This gives

$$Z_1 = 1 + \frac{\beta^2}{8\pi}\,\ell n\Big(\frac{\Lambda^2}{M^2}\Big). \tag{6.6}$$

Substituting (6.6) into (6.5) and keeping only the leading terms in β^2 we get

$$M_s = \frac{8}{\beta^2}\sqrt{\alpha_r(M^2)\Big[1 + \frac{\beta^2}{8\pi}\,\ell n\Big(\frac{\alpha_r(M^2)}{M^2}\Big)\Big]}. \tag{6.7}$$

Using Eq.(5.35) we can rewrite the r.h.s. of (6.7) in terms of $\alpha_{\rm ph}$:

$$M_s = \frac{8\sqrt{\alpha_{\rm ph}}}{\beta^2}. \tag{6.8}$$

The mass of a soliton M_s depends on the physical coupling constant $\alpha_{\rm ph}$. Hence, the contribution of Gaussian fluctuations around a soliton solution is absorbed by the renormalized coupling constant $\alpha_{\rm ph}$ and no singularities of the sine–Gordon model appear at $\beta^2 = 8\pi$.

This result confirms the assertion by Zamolodchikov and Zamolodchikov [8], that the singularity of the sine–Gordon model induced by the finite correction $-\sqrt{\alpha_{\rm ph}}/\pi$ to the soliton mass, caused by Gaussian fluctuations around a soliton solution, is completely due to the regularization and renormalization procedure. This has been corroborated in [2].

As has been shown in Section 3, a non–trivial finite correction to the soliton mass appears due to non–Gaussian quantum fluctuations of order of $\alpha^2_{\rm ph}\beta^2$ (see Eq.(3.31)):

$$M_s = \frac{8\sqrt{\alpha_{\rm ph}}}{\beta^2} - \frac{\sqrt{\alpha_{\rm ph}}}{\pi}. \tag{3.31}$$

The second term in Eq.(3.31) coincides with that obtained by Dashen *et al.* [6, 7]. However, it is meaningful only as a perturbative correction for $\beta^2 \ll 8\pi$, which cannot produce any non–perturbative singularity of the sine–Gordon model at $\beta^2 = 8\pi$.

We have obtained that the soliton mass M_s does not depend on the normalization scale M. This testifies that the soliton mass M_s is an observable quantity.

7 Renormalization of soliton mass by Gaussian fluctuations. Space–time discretization technique

Usually the correction to the soliton mass is investigated in the literature by a discretization procedure [14] (see also [15]). The soliton with Gaussian fluctuations is embedded

into a spatial box with a finite volume L and various boundary conditions for Gaussian fluctuations at $x = \pm L/2$. In such a discretization approach time is also discrete with a period T, which finally has to be taken in the limit $T \to \infty$. The frequency spectrum is $\omega_m = 2\pi m/T$ with $m = 0, \pm 1, \ldots$. For various boundary conditions spectra of the momenta of Gaussian fluctuations around a soliton and of Klein–Gordon quanta, corresponding to vacuum fluctuations, are adduced in Table 1. According to Table 1, for various boundary conditions the corrections to the soliton mass are given by

$$
\begin{aligned}
\Delta M_s^{(\rm p)} &= \lim_{T\to\infty}\lim_{L\to\infty}\frac{1}{2iT}\Big\{2\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\Big[\ell n\Big(\alpha_0 - \frac{4\pi^2 m^2}{T^2} + k^2_n\Big) - \ell n\Big(\alpha_0 - \frac{4\pi^2 m^2}{T^2} + q^2_n\Big)\Big]\\
&+ \sum_{m=-\infty}^{\infty}\Big[\ell n\Big(-\frac{4\pi^2 m^2}{T^2} + \Delta^2(L)\Big) - \ell n\Big(\alpha_0 - \frac{4\pi^2 m^2}{T^2}\Big)\Big]\Big\},\\
\Delta M_s^{(\rm ap)} &= \lim_{T\to\infty}\lim_{L\to\infty}\frac{1}{2iT}\Big\{2\sum_{m=-\infty}^{\infty}\sum_{n=2}^{\infty}\Big[\ell n\Big(\alpha_0 - \frac{4\pi^2 m^2}{T^2} + k^2_n\Big) - \ell n\Big(\alpha_0 - \frac{4\pi^2 m^2}{T^2} + q^2_n\Big)\Big]\\
&+ \sum_{m=-\infty}^{\infty}\Big[\ell n\Big(-\frac{4\pi^2 m^2}{T^2} + \Delta^2(L)\Big) + \ell n\Big(\alpha_0 - \frac{4\pi^2 m^2}{T^2}\Big) - 2\ell n\Big(\alpha_0 - \frac{4\pi^2 m^2}{T^2} + q^2_1\Big)\Big]\Big\},\\
\Delta M_s^{(\rm rw)} &= \lim_{T\to\infty}\lim_{L\to\infty}\frac{1}{2iT}\Big\{\sum_{m=-\infty}^{\infty}\sum_{n=2}^{\infty}\Big[\ell n\Big(\alpha_0 - \frac{4\pi^2 m^2}{T^2} + k^2_n\Big) - \ell n\Big(\alpha_0 - \frac{4\pi^2 m^2}{T^2} + q^2_n\Big)\Big]\\
&+ \sum_{m=-\infty}^{\infty}\Big[\ell n\Big(-\frac{4\pi^2 m^2}{T^2} + \Delta^2(L)\Big) - \ell n\Big(\alpha_0 - \frac{4\pi^2 m^2}{T^2} + q^2_1\Big)\Big]\Big\},
\end{aligned}
\tag{7.1}
$$

where $\Delta^2(L) \sim \alpha_0\, e^{-\sqrt{\alpha_0}\,L}$ at $L \to \infty$, and the abbreviations (p), (ap) and (rw) mean periodic, anti–periodic boundary conditions and rigid walls, respectively.

For the summation over m we use the formula, derived by Dolan and Jackiw [16]:

$$
\sum_{m=-\infty}^{+\infty}\Big[\ell n\Big(\frac{4m^2\pi^2}{-T^2} + a^2\Big) - \ell n\Big(\frac{4m^2\pi^2}{-T^2} + b^2\Big)\Big] = i\,(a-b)\,T + 2\,\ell n\Big(\frac{1 - e^{-i\,a\,T}}{1 - e^{-i\,b\,T}}\Big). \tag{7.2}
$$

Taking the limit $T \to \infty$ we arrive at the expression

$$
\Delta M_s = -\frac{\sqrt{\alpha_0}}{2} + \lim_{L\to\infty}\left\{
\begin{array}{lcl}
\displaystyle\sum_{n=1}^{\infty}\big(\sqrt{\alpha_0 + k^2_n} - \sqrt{\alpha_0 + q^2_n}\,\big) & , & \text{periodic BC}\\
\displaystyle\sum_{n=2}^{\infty}\big(\sqrt{\alpha_0 + k^2_n} - \sqrt{\alpha_0 + q^2_n}\,\big) & , & \text{anti}-\text{periodic BC}\\
\displaystyle\frac{1}{2}\sum_{n=2}^{\infty}\big(\sqrt{\alpha_0 + k^2_n} - \sqrt{\alpha_0 + q^2_n}\,\big) & , & \text{rigid walls,}
\end{array}\right. \tag{7.3}
$$

where BC is the abbreviation of *boundary conditions*.

The aim of our analysis of ΔM_s within the discretization procedure is to show that the discretization procedure gives ΔM_s in the form of (6.3).

The subsequent analysis of ΔM_s we carry out for periodic boundary conditions only. One can show that for anti–periodic boundary conditions and rigid walls the result is the same.

Table 1: The spectra of the momenta of Gaussian fluctuations around a soliton and the Klein–Gordon quanta. The modes, denoted by $(*)$ are due to the bound state.

PERIODIC BC:

Soliton Sector		**Vacuum Sector**
$k_n L + \delta(k_n) = 2n\pi$		$q_n L = 2n\pi$
$n = 0 : (*)$	$\leftarrow 1\times \rightarrow$	$n = 0 : q_0 = 0$
$n = 1 : q_1 = \frac{\pi}{L} + \mathcal{O}(L^{-2})$	$\leftarrow 2\times \rightarrow$	$n = 1 : q_1 = \frac{2\pi}{L}$
$\cdots$	$\leftarrow 2\times \rightarrow$	$\cdots$
$\sum_{n=1}^{\infty} \rightarrow \int_{\frac{\pi}{L}+\mathcal{O}(L^{-2})}^{\infty} \frac{dk}{2\pi}\left(L + \frac{d\delta(k)}{dk}\right)$		$\sum_{n=1}^{\infty} \rightarrow \int_{\frac{2\pi}{L}}^{\infty} \frac{dq}{2\pi} L$

ANTI-PERIODIC BC:

Soliton Sector		**Vacuum Sector**
$k_n L + \delta(k_n) = (2n-1)\pi$		$q_n L = (2n-1)\pi$
$n = 1 : (*) + k_1 = 0$	$\leftarrow (1+1)\times \rightarrow$	$n = 1 : q_1 = \frac{\pi}{L}$
$\cdots$	$\leftarrow 2\times \rightarrow$	$\cdots$
$\sum_{n=2}^{\infty} \rightarrow \int_{\frac{2\pi}{L}+\mathcal{O}(L^{-2})}^{\infty} \frac{dk}{2\pi}\left(L + \frac{d\delta(k)}{dk}\right)$		$\sum_{n=2}^{\infty} \rightarrow \int_{\frac{3\pi}{L}}^{\infty} \frac{dq}{2\pi} L$

RIGID WALLS:

Soliton Sector		**Vacuum Sector**
$k_n L + \delta(k_n) = n\pi$		$q_n L = n\pi$
$n = 1 : (*)$	$\leftarrow 1\times \rightarrow$	$n = 1 : q_1 = \frac{\pi}{L}$
$\cdots$	$\leftarrow 1\times \rightarrow$	$\cdots$
$\sum_{n=2}^{\infty} \rightarrow \int_{\frac{\pi}{L}+\mathcal{O}(L^{-2})}^{\infty} \frac{dk}{\pi}\left(L + \frac{d\delta(k)}{dk}\right)$		$\sum_{n=2}^{\infty} \rightarrow \int_{\frac{2\pi}{L}}^{\infty} \frac{dq}{\pi} L$

For the next transformation of the r.h.s. of (7.3) we propose to use the following integral representation

$$\sqrt{\alpha_0 + k_n^2} - \sqrt{\alpha_0 + q_n^2} = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i}\, \ell n\Big[\frac{\alpha_0 + k_n^2 - \omega^2 - i0}{\alpha_0 + q_n^2 - \omega^2 - i0}\Big]. \tag{7.4}$$

This gives

$$\begin{aligned}\Delta M_s^{(\mathrm{p})} &= -\frac{\sqrt{\alpha_0}}{2} + \lim_{L\to\infty}\sum_{n=1}^{+\infty}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi i}\,\ell n\Big[\frac{\alpha_0 + k_n^2 - \omega^2 - i\,0}{\alpha_0 + q_n^2 - \omega^2 - i\,0}\Big] = \\ &= -\frac{\sqrt{\alpha_0}}{2} + \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi i}\lim_{L\to\infty}\sum_{n=1}^{+\infty}\ell n\Big[\frac{\alpha_0 + k_n^2 - \omega^2 - i\,0}{\alpha_0 + q_n^2 - \omega^2 - i\,0}\Big].\end{aligned} \tag{7.5}$$

To the regularization of the sum over n we apply the *mode–counting* regularization procedure [15]:

$$\Delta M_s^{(\mathrm{p})} = -\frac{\sqrt{\alpha_0}}{2} + \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi i}\lim_{L\to\infty}\lim_{N\to\infty}\sum_{n=1}^{N}\ell n\Big[\frac{\alpha_0 + k_n^2 - \omega^2 - i\,0}{\alpha_0 + q_n^2 - \omega^2 - i\,0}\Big]. \tag{7.6}$$

Passing to the continuous momentum representation [15] and using the spectra of the momenta of Gaussian fluctuations and the Klein–Gordon fluctuations (the vacuum fluctuations) adduced in Table 1 we transcribe the r.h.s of (7.6) into the form

$$\begin{aligned}\Delta M_s^{(\mathrm{p})} &= -\frac{\sqrt{\alpha_0}}{2} + \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi i}\lim_{L\to\infty}\lim_{N\to\infty}\Big[\int_{k_1}^{k_N} dk\,\frac{dn(k)}{dk}\,\ell n(\alpha_0 + k^2 - \omega^2 - i\,0) \\ &\quad - \int_{q_1}^{q_N} dq\,\frac{dn(q)}{dq}\ell n(\alpha_0 + q^2 - \omega^2 - i\,0)\Big] = \\ &\quad -\frac{\sqrt{\alpha_0}}{2} + \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi i}\lim_{L\to\infty}\lim_{N\to\infty}\Big[\int_{k_1}^{k_N}\frac{dk}{2\pi}\Big(L + \frac{d\delta(k)}{dk}\Big) \\ &\quad \times \ell n(\alpha_0 + k^2 - \omega^2 - i\,0) - L\int_{q_1}^{q_N}\frac{dk}{2\pi}\,\ell n(\alpha_0 + k^2 - \omega^2 - i\,0)\Big],\end{aligned} \tag{7.7}$$

where the limits are equal to (see Table 1):

$$\begin{aligned} k_1 &= \frac{\pi}{L} \quad , \quad k_N = q_N - \frac{\delta(q_N)}{L} = q_N - \frac{\sqrt{\alpha_0}}{\pi N}, \\ q_1 &= \frac{2\pi}{L} \quad , \quad q_N = \frac{2\pi N}{L}. \end{aligned} \tag{7.8}$$

Rearranging the limits of integrations we get

$$\begin{aligned}\Delta M_s^{(\mathrm{p})} &= -\frac{\sqrt{\alpha_0}}{2} + \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi i}\lim_{L\to\infty}\lim_{N\to\infty}\int_{\pi/L}^{k_N}\frac{dk}{2\pi}\,\frac{d\delta(k)}{dk}\,\ell n(\alpha_0 + k^2 - \omega^2 - i\,0) \\ &\quad + \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi i}\lim_{L\to\infty} L\int_{\pi/L}^{2\pi/L}\frac{dk}{2\pi}\,\ell n(\alpha_0 + k^2 - \omega^2 - i\,0) \\ &\quad - \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi i}\lim_{L\to\infty}\lim_{N\to\infty} L\int_{k_N}^{q_N}\frac{dk}{2\pi}\,\ell n(\alpha_0 + k^2 - \omega^2 - i\,0) =\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\sqrt{\alpha_0}}{2} + \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \lim_{L\to\infty} \int_{\pi/L}^{\infty} \frac{dk}{2\pi} \frac{d\delta(k)}{dk}\, \ell n(\alpha_0 + k^2 - \omega^2 - i\,0) \\
&+ \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \lim_{L\to\infty} L \int_{\pi/L}^{2\pi/L} \frac{dk}{2\pi}\, \ell n(\alpha_0 + k^2 - \omega^2 - i\,0) \\
&- \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \lim_{L\to\infty} \lim_{N\to\infty} L\, \frac{k_N - q_N}{2\pi}\, \ell n(\alpha_0 + k^{*\,2} - \omega^2 - i\,0). \qquad (7.9)
\end{aligned}
$$

For the last term we have applied the *mean value theorem* with $q_N - \delta(q_N)/L < k^* < q_N$. Since the difference $k_N - q_N = \delta(q_N)/L = \sqrt{\alpha_0}/\pi N$ is of order $O(1/N)$, this term vanishes in the limit $N \to \infty$ [6]. As a result we get

$$
\begin{aligned}
\Delta M_s^{(\mathrm{p})} &= -\frac{\sqrt{\alpha_0}}{2} + \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \lim_{L\to\infty} \int_{\pi/L}^{\infty} \frac{dk}{2\pi} \frac{d\delta(k)}{dk}\, \ell n(\alpha_0 + k^2 - \omega^2 - i\,0) \\
&+ \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \lim_{L\to\infty} L \int_{\pi/L}^{2\pi/L} \frac{dk}{2\pi}\, \ell n(\alpha_0 + k^2 - \omega^2 - i\,0). \qquad (7.10)
\end{aligned}
$$

Taking the limit $L \to \infty$ and applying to the last term the *mean value theorem* we arrive at the expression

$$
\begin{aligned}
\Delta M_s^{(\mathrm{p})} &= -\frac{\sqrt{\alpha_0}}{2} + \int_{-\infty}^{+\infty} \frac{d\omega}{4\pi i}\, \ell n(\alpha_0 - \omega^2 - i\,0) \\
&+ \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \int_{0}^{\infty} \frac{dk}{2\pi} \frac{d\delta(k)}{dk}\, \ell n(\alpha_0 + k^2 - \omega^2 - i\,0), \qquad (7.11)
\end{aligned}
$$

which we transcribe into the form

$$
\begin{aligned}
\Delta M_s^{(\mathrm{p})} &= -\frac{\sqrt{\alpha_0}}{2} + \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \int_{0}^{\infty} \frac{dk}{2\pi} \frac{d\delta(k)}{dk}\, \ell n(-\omega^2 - i\,0) + \int_{-\infty}^{+\infty} \frac{d\omega}{4\pi i}\, \ell n(\alpha_0 - \omega^2 - i\,0) \\
&+ \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \int_{0}^{\infty} \frac{dk}{2\pi} \frac{d\delta(k)}{dk} \left[\ell n(\alpha_0 + k^2 - \omega^2 - i\,0) - \ell n(-\omega^2 - i\,0)\right]. \qquad (7.12)
\end{aligned}
$$

The next steps of the reduction of the r.h.s. of (7.12) to the form (6.3) are rather straightforward. First, one can easily show that

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \int_{0}^{\infty} \frac{dk}{2\pi} \frac{d\delta(k)}{dk}\, \ell n(-\omega^2 - i\,0) + \int_{-\infty}^{+\infty} \frac{d\omega}{4\pi i}\, \ell n(\alpha_0 - \omega^2 - i\,0) = \\
&= \int_{-\infty}^{+\infty} \frac{d\omega}{4\pi i} \left[\ell n(\alpha_0 - \omega^2 - i\,0) - \ell n(-\omega^2 - i\,0)\right] = \frac{\sqrt{\alpha_0}}{2} \qquad (7.13)
\end{aligned}
$$

and, second, integrating over ω by parts the last integral in (7.12) can be reduced to the form

$$
\Delta M_s^{(\mathrm{p})} = -2\sqrt{\alpha_0} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \frac{1}{\alpha_0 - \omega^2 + k^2 - i\,0}. \qquad (7.14)
$$

[6] We would like to emphasize that exactly the term of this kind leads to the contribution of the finite *surface term* $-\sqrt{\alpha_0}/\pi$ in a regularization procedure using the expressions (7.3) without turning to the integral representation (7.4).

Thus, we have shown that the correction to the soliton mass, induced by Gaussian fluctuations around a soliton and calculated by means of the discretization procedure, agrees fully with that we have obtained in continuous space–time (6.3).

Hence, the renormalization of the soliton mass, caused by Gaussian fluctuations calculated within the space–time discretization technique, coincides with the renormalization of the soliton mass in continuous space–time. We would like to emphasize that the obtained result (7.14) does not depend on the boundary conditions.

The calculation of the functional determinant within the discretization procedure has confirmed the absence of the correction $-\sqrt{\alpha_0}/\pi$. This agrees with our assertion that such a correction does not appear due to Gaussian fluctuations around a soliton, corresponding to quantum fluctuations to first orders in $\alpha_0(\Lambda^2)$ and β^2.

The reduction of $\Delta M^{(\rm p)}_s$ of Eq.(7.1) to the expression (6.3) can be carried out directly. First, summing over n within the *mode–counting* regularization procedure and taking the limit $L \to \infty$ we arrive from $\Delta M^{(\rm p)}_s$ of Eq.(7.1) at

$$\begin{aligned}
\Delta M^{(\rm p)}_s &= \lim_{T\to\infty}\frac{1}{iT}\sum^{\infty}_{m=-\infty}\lim_{L\to\infty}\Big\{\frac{1}{2}\Big[\ell n\Big(-\frac{4\pi^2m^2}{T^2}+\Delta^2(L)\Big)-\ell n\Big(\alpha_0-\frac{4\pi^2m^2}{T^2}\Big)\Big]\\
&+\lim_{N\to\infty}\sum^{N}_{n=1}\Big[\ell n\Big(\alpha_0+k^2_n-\frac{4\pi^2m^2}{T^2}\Big)-\ell n\Big(\alpha_0+q^2_n-\frac{4\pi^2m^2}{T^2}\Big)\Big]\Big\}=\\
&=\lim_{T\to\infty}\frac{1}{iT}\sum^{\infty}_{m=-\infty}\lim_{L\to\infty}\Big\{\frac{1}{2}\Big[\ell n\Big(-\frac{4\pi^2m^2}{T^2}+\Delta^2(L)\Big)-\ell n\Big(\alpha_0-\frac{4\pi^2m^2}{T^2}\Big)\Big]\\
&+\lim_{N\to\infty}\Big[\int^{k_N}_{k_1}dk\frac{dn(k)}{dk}\ell n\Big(\alpha_0+k^2-\frac{4\pi^2m^2}{T^2}\Big)-\int^{q_N}_{q_1}dq\frac{dn(q)}{dq}\ell n\Big(\alpha_0+q^2-\frac{4\pi^2m^2}{T^2}\Big)\Big]\Big\}=\\
&=\lim_{T\to\infty}\frac{1}{iT}\sum^{\infty}_{m=-\infty}\lim_{L\to\infty}\Big\{\frac{1}{2}\Big[\ell n\Big(-\frac{4\pi^2m^2}{T^2}+\Delta^2(L)\Big)-\ell n\Big(\alpha_0-\frac{4\pi^2m^2}{T^2}\Big)\Big]\\
&+\lim_{N\to\infty}\Big[\int^{k_N}_{\pi/L}\frac{dk}{2\pi}\frac{d\delta(k)}{dk}\,\ell n\Big(\alpha_0+k^2-\frac{4\pi^2m^2}{T^2}\Big)+L\int^{2\pi/L}_{\pi/L}\frac{dk}{2\pi}\,\ell n\Big(\alpha_0+k^2-\frac{4\pi^2m^2}{T^2}\Big)\\
&-L\int^{q_N}_{k_N}\frac{dk}{2\pi}\,\ell n\Big(\alpha_0-\frac{4\pi^2m^2}{T^2}+k^2\Big)\Big]=\\
&=\lim_{T\to\infty}\frac{1}{iT}\sum^{\infty}_{m=-\infty}\lim_{L\to\infty}\Big\{\frac{1}{2}\Big[\ell n\Big(-\frac{4\pi^2m^2}{T^2}+\Delta^2(L)\Big)-\ell n\Big(\alpha_0-\frac{4\pi^2m^2}{T^2}\Big)\Big]\\
&+\int^{\infty}_{\pi/L}\frac{dk}{2\pi}\frac{d\delta(k)}{dk}\,\ell n\Big(\alpha_0+k^2-\frac{4\pi^2m^2}{T^2}\Big)+L\int^{2\pi/L}_{\pi/L}\frac{dk}{2\pi}\,\ell n\Big(\alpha_0+k^2-\frac{4\pi^2m^2}{T^2}\Big)\Big\}=\\
&=\lim_{T\to\infty}\frac{1}{iT}\sum^{\infty}_{m=-\infty}\Big\{\frac{1}{2}\Big[\ell n\Big(-\frac{4\pi^2m^2}{T^2}\Big)-\ell n\Big(\alpha_0-\frac{4\pi^2m^2}{T^2}\Big)\Big]\\
&+\int^{\infty}_{0}\frac{dk}{2\pi}\frac{d\delta(k)}{dk}\,\ell n\Big(\alpha_0+k^2-\frac{4\pi^2m^2}{T^2}\Big)+\frac{1}{2}\,\ell n\Big(\alpha_0-\frac{4\pi^2m^2}{T^2}\Big)\Big\}=\\
&=\lim_{T\to\infty}\frac{1}{iT}\sum^{\infty}_{m=-\infty}\Big\{\int^{\infty}_{0}\frac{dk}{2\pi}\frac{d\delta(k)}{dk}\,\ell n\Big(\alpha_0+k^2-\frac{4\pi^2m^2}{T^2}\Big)+\frac{1}{2}\,\ell n\Big(-\frac{4\pi^2m^2}{T^2}\Big)\Big\}. \qquad (7.15)
\end{aligned}$$

Now we use the integral representation [5] and get

$$\Delta M_S^{(\mathrm{p})} = \lim_{T\to\infty}\frac{1}{iT}\sum_{m=-\infty}^{\infty}\int_0^{\infty}\frac{dk}{2\pi}\,\frac{2\sqrt{\alpha_0}}{\alpha_0+k^2}\,\Big[\ell n\Big(-\frac{4\pi^2m^2}{T^2}\Big)-\ell n\Big(\alpha_0+k^2-\frac{4\pi^2m^2}{T^2}\Big)\Big] =$$

$$= \lim_{T\to\infty}\int_{-\infty}^{+\infty}\frac{d\omega}{iT}\,\frac{dm(\omega)}{d\omega}\int_0^{\infty}\frac{dk}{2\pi}\,\frac{2\sqrt{\alpha_0}}{\alpha_0+k^2}\,[\ell n(-\omega^2)-\ell n(\alpha_0+k^2-\omega^2)] =$$

$$= \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi i}\int_0^{\infty}\frac{dk}{2\pi}\,\frac{2\sqrt{\alpha_0}}{\alpha_0+k^2}\,[\ell n(-\omega^2)-\ell n(\alpha_0+k^2-\omega^2)] =$$

$$= -2\sqrt{\alpha_0}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi i}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\frac{1}{\alpha_0-\omega^2+k^2-i\,0}. \tag{7.16}$$

For the other boundary conditions we get the same result.

Thus, we have shown that the discretized version of the correction to the soliton mass reduces to the continuum result if one transcribes first the sum over the quantum number n of the momenta of Gaussian and vacuum fluctuations into the corresponding integral over the momentum k.

8 Conclusion

We have investigated the renormalizability of the sine–Gordon model. We have analysed the renormalizability of the two–point Green function to second order in α and to all orders in β^2. We have shown that the divergences appearing in the sine–Gordon model can be removed by the renormalization of the dimensional coupling constant $\alpha_0(\Lambda^2)$. We remind that the coupling constant β^2 is not renormalizable. This agrees well with a possible interpretation of the coupling constant β^2 as $\hbar$ [1, 17]. The quantum fluctuations calculated to first order in $\alpha_r(M^2)$, the renormalized dimensional coupling constant depending on the normalization scale M, and to arbitrary order in β^2 after removal of divergences form a physical coupling constant $\alpha_{\rm ph}$, which is finite and does not depend on the normalization scale M. We have argued that the total renormalized two–point Green function depends on the physical coupling constant $\alpha_{\rm ph}$ only. In order to illustrate this assertion (i) we have calculated the correction to the two–point Green function to second order in $\alpha_r(M^2)$ and to all orders in β^2 and (ii) we have solved the Callan–Symanzik equation for the two–point Green function with the Gell–Mann–Low function, defined to all orders in $\alpha_r(M^2)$ and β^2. We have found that the two–point Green function of the sine–Gordon field depends on the running coupling constant $\alpha_r(p^2) = \alpha_{\rm ph}(p^2/\alpha_{\rm ph})^{\tilde{\beta}^2/8\pi}$, where $\tilde{\beta}^2 = \beta^2/(1+\beta^2/8\pi) < 1$ for any β^2.

We have shown that the finite contribution of the quantum fluctuations, calculated to second order in $\alpha_{\rm ph}$ and to first order in β^2, leads to an effective Lagrangian

$$\mathcal{L}(x) = \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha_{\rm eff}}{\beta^2}\,(\cos\beta\vartheta(x)-1) \tag{3.30}$$

with the effective coupling constant $\alpha_{\rm eff} = \alpha_{\rm ph}\,(1-\beta^2/4\pi)$, which defines the soliton mass

$$M_s = \frac{8\sqrt{\alpha_{\rm eff}}}{\beta^2} = \frac{8\sqrt{\alpha_{\rm ph}}}{\beta^2} - \frac{\sqrt{\alpha_{\rm ph}}}{\pi}. \tag{3.31}$$

The term $-\sqrt{\alpha_{\rm ph}}/\pi$ is a perturbative finite correction to the soliton mass, which coincides with the result obtained by Dashen *et al.* [6, 7]. Such a correction has been interpreted in the literature as a non–perturbative contribution leading to a singularity of the sine–Gordon model at $\beta^2 = 8\pi$ (see also [9]). However, in our approach such a correction is a perturbative one valid for $\beta^2 \ll 8\pi$ and does not provide any singularity for the sine–Gordon model in the non–perturbative regime $\beta^2 \sim 8\pi$. This confirms the conjecture by Zamolodchikov and Zamolodchikov [8] that a singularity of the sine–Gordon model at $\beta^2 = 8\pi$ [6, 7, 9] is a superficial one and depends on the regularization and renormalization procedure. This has been corroborated in [2].

In addition to the analysis of the renormalizability of the sine–Gordon model with respect to quantum fluctuations relative to the trivial vacuum, we have analysed the renormalizability of the sine–Gordon model with respect to quantum fluctuations around a soliton. Following Dashen *et al.* [6, 7] and Faddeev and Korepin [9] we have taken into account only Gaussian fluctuations.

For the calculation of the effective Lagrangian, induced by Gaussian fluctuations, we have used the path–integral approach and integrated over the field $\varphi(x)$, fluctuating around a soliton. This has allowed to express the effective Lagrangian in terms of the functional determinant. For the calculation of the contribution of the functional determinant we have used the eigenfunctions and eigenvalues of the differential operator, describing the evolution of the field $\varphi(x)$. We have shown that the renormalized effective Lagrangian, induced by Gaussian fluctuations around a soliton, coincides completely with the renormalized Lagrangian of the sine–Gordon model, caused by quantum fluctuations around the trivial vacuum to first order in α_0 and to second order in β^2. After the removal of divergences the total effective Lagrangian, caused by Gaussian fluctuations around a soliton, is equal to (3.21)

$$\mathcal{L}(x) = \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha_{\rm ph}}{\beta^2}\,(\cos\beta\vartheta(x) - 1). \tag{3.21}$$

This implies that Gaussian fluctuations around a soliton do not produce any quantum corrections to the soliton mass. After the removal divergences the soliton mass is equal to the mass of a soliton, calculated without quantum corrections, with the replacement $\alpha_0 \to \alpha_{\rm ph}$. Hence, no non–perturbative singularities of the sine Gordon model at $\beta^2 = 8\pi$ can be induced by Gaussian fluctuations around a soliton.

For the confirmation of our results, obtained in continuous space–time, we have calculated the functional determinant caused by Gaussian fluctuations around a soliton within the discretization procedure with periodic and anti–periodic boundary conditions and rigid walls. We have shown that the result of the calculation of the functional determinant (i) coincides with that we have obtained in continuous space–time and (ii) does not depend on the boundary conditions.

Appendix. Solutions of the differential equation for Gaussian fluctuations around a soliton

Making a change of variables $\xi = \tanh(\sqrt{\alpha_0}\sigma)$ we reduce (5.15) to the form [18]

$$\frac{d}{d\xi}\Big[(1-\xi^2)\frac{d\psi(\xi)}{d\xi}\Big] + \Big[s(s+1) - \frac{\epsilon^2}{1-\xi^2}\Big]\psi(\xi) = 0, \tag{A.1}$$

where $s(s+1) = 2$ and $\epsilon^2 = -k^2/\alpha_0 = 1 - \omega^2/\alpha_0$ [18].

Substituting $\psi(\xi) = (1-\xi^2)^{\epsilon/2}\, w(\xi)$ and denoting $u = (1-\xi)/2$ we arrive at the equation [18]

$$u(1-u)w'' + (\epsilon+1)(1-2u)\, w' - (\epsilon - s)(\epsilon + s + 1)\, w = 0. \tag{A.2}$$

The solution of this equation can be given in terms of hypergeometric functions $F(a,b;c;z)$ [19] (see also [20]). It reads

$$w(\xi) = w^{(1)}(\xi) + w^{(2)}(\xi) = C_1\, F\Big(\epsilon - s, \epsilon + s + 1; \epsilon + 1; \frac{1-\xi}{2}\Big)$$
$$+C_2\,\Big(\frac{1-\xi}{2}\Big)^{-\epsilon} F\Big(-s, s+1; 1-\epsilon; \frac{1-\xi}{2}\Big), \tag{A.3}$$

where C_1 and C_2 are the integration constants.

The parameter s acquires two values $s = -2, +1$, which are solutions of the equation $s(s+1) = 2$. Since for both case the hypergeometric functions coincide [19], setting $s = 1$ we obtain

$$w(\xi) = w^{(1)}(\xi) + w^{(2)}(\xi) = C_1\, F\Big(\epsilon - 1, \epsilon + 2; \epsilon + 1; \frac{1-\xi}{2}\Big)$$
$$+C_2\,\Big(\frac{1-\xi}{2}\Big)^{-\epsilon} F\Big(-1, 2; 1-\epsilon; \frac{1-\xi}{2}\Big). \tag{A.4}$$

For arbitrary ϵ the solution $w^{(2)}(\xi)$ can be reduced to the polynomial [19]

$$w^{(2)}(\xi) = C_2\,\Big(\frac{1-\xi}{2}\Big)^{-\epsilon} F\Big(2, -1; 1-\epsilon; \frac{1-\xi}{2}\Big) =$$
$$= C_2\,\Big(\frac{1-\xi}{2}\Big)^{-\epsilon} F\Big(-1, 2; 1-\epsilon; \frac{1-\xi}{2}\Big) = C_2\,\Big(\frac{1-\xi}{2}\Big)^{-\epsilon}\,\frac{\xi-\epsilon}{1-\epsilon}. \tag{A.5}$$

One can show that the solution $w^{(1)}(\xi)$ can be reduced to a polynomial too. For this aim we suggest to use the following relation for hypergeometric functions [20, 19]

$$F\Big(a, b; c; \frac{1-\xi}{2}\Big) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\, F\Big(a, b; a+b-c+1; \frac{1+\xi}{2}\Big)$$
$$+\Big(\frac{1+\xi}{2}\Big)^{c-a-b}\,\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\, F\Big(c-a, c-b; c-a-b+1; \frac{1+\xi}{2}\Big). \tag{A.6}$$

For the solution $w^{(1)}(\xi)$ we get

$$w^{(1)}(\xi) = C_1\, F\Big(\epsilon-1,\epsilon+2;\epsilon+1;\frac{1-\xi}{2}\Big) = C_1\,\frac{\Gamma(\epsilon+1)\Gamma(-\epsilon)}{\Gamma(-1)}\,F\Big(\epsilon-1,\epsilon+2;\epsilon+1;\frac{1+\xi}{2}\Big)$$

$$+C_1\,\frac{\Gamma(\epsilon+1)\Gamma(\epsilon)}{\Gamma(\epsilon-1)\Gamma(\epsilon+2)}\,\Big(\frac{1+\xi}{2}\Big)^{-\epsilon}\,F\Big(2,-1;1-\epsilon;\frac{1+\xi}{2}\Big) = C_1\,\Big(\frac{1+\xi}{2}\Big)^{-\epsilon}\,\frac{\epsilon+\xi}{1+\epsilon}. \qquad (A.7)$$

Thus, the fluctuating field $\psi(\xi)$ is equal to

$$\psi(\xi) = C_1\,2^{\epsilon}\,\Big(\frac{1-\xi}{1+\xi}\Big)^{\epsilon/2}\,\frac{\epsilon+\xi}{1+\epsilon} + C_2\,2^{\epsilon}\,\Big(\frac{1+\xi}{1-\xi}\Big)^{\epsilon/2}\,\frac{\xi-\epsilon}{1-\epsilon}. \qquad (A.8)$$

In terms of σ the classical solution for the fluctuating field reads

$$\psi(\sigma) = C_1\,(\epsilon+\tanh(\sqrt{\alpha_0}\sigma))\,e^{\,-\epsilon\sqrt{\alpha_0}\sigma} + C_2\,(-\epsilon+\tanh(\sqrt{\alpha_0}\sigma))\,e^{\,+\epsilon\sqrt{\alpha_0}\sigma}, \qquad (A.9)$$

where we have redefined the integration constants. These solutions describe a bound state for $\epsilon = \pm 1$ and a scattering state for $\epsilon = \pm ik/\sqrt{\alpha_0}$. They are

$$\psi_b(\sigma) = \sqrt{\frac{\sqrt{\alpha_0}}{2}}\,\frac{1}{\cosh(\sqrt{\alpha_0}\sigma)},$$

$$\psi_k(\sigma) = \frac{i}{\sqrt{2\pi}}\,\frac{-ik+\sqrt{\alpha_0}\,\tanh(\sqrt{\alpha_0}\sigma)}{\sqrt{k^2+\alpha_0}}\,e^{\,+ik\sigma}. \qquad (A.10)$$

The wave functions $\psi_k(\sigma)$, given by (A.10), are normalized to the δ–function as

$$\int^{+\infty}_{-\infty} d\sigma\,\psi^*_{k\,'}(\sigma)\psi_k(\sigma) = \frac{1}{k\,'^{\,2}-k^2}\,\Big(\psi^*_{k\,'}(\sigma)\frac{d}{d\sigma}\psi_k(\sigma) - \psi_k(\sigma)\frac{d}{d\sigma}\psi^*_{k\,'}(\sigma)\Big)\Big|^{+\infty}_{-\infty} = \delta(k\,'-k). \qquad (A.11)$$

The fluctuating field $\varphi(\tau,\sigma)$ is equal to

$$\varphi_{\omega b}(\tau,\sigma) = \frac{1}{\sqrt{2\pi}}\,\sqrt{\frac{\sqrt{\alpha_0}}{2}}\,\frac{1}{\cosh(\sqrt{\alpha_0}\sigma)}\,e^{\,-i\omega\tau},$$

$$\varphi_{\omega k}(\tau,\sigma) = \frac{i}{2\pi}\,\frac{-ik+\sqrt{\alpha_0}\,\tanh(\sqrt{\alpha_0}\sigma)}{\sqrt{k^2+\alpha_0}}\,e^{\,-i\omega\tau\,+\,ik\sigma}. \qquad (A.12)$$

They are normalized by

$$\int^{+\infty}_{-\infty}\int^{+\infty}_{-\infty} d\tau\varphi^*_{\omega\,'b}(\tau,\sigma)\varphi_{\omega b}(\tau,\sigma) = \delta(\omega\,'-\omega),$$

$$\int^{+\infty}_{-\infty}\int^{+\infty}_{-\infty} d\tau d\sigma\,\varphi^*_{\omega\,'k\,'}(\tau,\sigma)\varphi_{\omega k}(\tau,\sigma) = \delta(\omega\,'-\omega)\delta(k\,'-k). \qquad (A.13)$$

The eigenfunction $\psi_b(\sigma)$ has the eigenvalue zero, $\omega = 0$. This corresponds to the bound state [11]. The eigenfunction $\psi_b(\sigma)$ is normalized to unity

$$\int^{+\infty}_{-\infty} d\sigma\,|\psi_b(\sigma)|^2 = 1. \qquad (A.14)$$

The solutions (A.10) satisfy the completeness condition [11]

$$\int_{-\infty}^{+\infty} dk\,\psi^*_k(\sigma\,')\psi_k(\sigma) + \psi_b(\sigma\,')\psi_b(\sigma) = \delta(\sigma\,' - \sigma). \tag{A.15}$$

The proof of the completeness condition (A.15)

$$\int_{-\infty}^{+\infty} dk\,\psi^*_k(\sigma\,')\psi_k(\sigma) + \psi_b(\sigma\,')\psi_b(\sigma) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\,e^{ik(\sigma - \sigma\,')}$$
$$+\sqrt{\alpha_0}\,[\tanh(\sqrt{\alpha_0}\sigma) - \tanh(\sqrt{\alpha_0}\sigma\,')]\int_{-\infty}^{+\infty} \frac{dk}{2\pi}\,\frac{ik}{k^2+\alpha_0}\,e^{ik(\sigma - \sigma\,')}$$
$$+\alpha_0\,[\tanh(\sqrt{\alpha_0}\sigma\,')\tanh(\sqrt{\alpha_0}\sigma) - 1]\int_{-\infty}^{+\infty} \frac{dk}{2\pi}\,\frac{1}{k^2+\alpha_0}\,e^{ik(\sigma - \sigma\,')}$$
$$+\frac{\sqrt{\alpha_0}}{2}\,\frac{1}{\cosh(\sqrt{\alpha_0}\sigma\,')}\,\frac{1}{\cosh(\sqrt{\alpha_0}\sigma)}. \tag{A.16}$$

Integrating over k we get

$$\int_{-\infty}^{+\infty} dk\,\psi^*_k(\sigma\,')\psi_k(\sigma) + \psi_b(\sigma\,')\psi_b(\sigma) = \delta(\sigma\,' - \sigma) - \frac{\sqrt{\alpha_0}}{2}\,\frac{1}{\cosh(\sqrt{\alpha_0}\sigma\,')}\,\frac{1}{\cosh(\sqrt{\alpha_0}\sigma)}$$
$$\times\, e^{-\sqrt{\alpha_0}|\sigma - \sigma\,'|}\,[\varepsilon(\sigma - \sigma\,')\,\sinh(\sqrt{\alpha_0}(\sigma - \sigma\,')) + \cosh(\sqrt{\alpha_0}(\sigma - \sigma\,'))]$$
$$+\frac{\sqrt{\alpha_0}}{2}\,\frac{1}{\cosh(\sqrt{\alpha_0}\sigma\,')}\,\frac{1}{\cosh(\sqrt{\alpha_0}\sigma)} = \delta(\sigma\,' - \sigma), \tag{A.17}$$

where $\varepsilon(\sigma - \sigma\,')$ is the sign–function. The term, proportional to $e^{-\sqrt{\alpha_0}|\sigma - \sigma\,'|}$, is given by the contributions of the second and the third terms of the r.h.s. of (A.16). For the derivation of (A.17) we have used the relation

$$\varepsilon(\sigma - \sigma\,')\,\sinh(\sqrt{\alpha_0}((\sigma - \sigma\,')) + \cosh(\sqrt{\alpha_0}(\sigma - \sigma\,')) = e^{+\sqrt{\alpha_0}|\sigma - \sigma\,'|}. \tag{A.17}$$

Thus, the contribution of the second and the third terms in the r.h.s. of (A.16) cancel the contribution of the zero–mode $\psi_b(\sigma)$. This completes the proof of the completeness condition (A.15).

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